Higher-order effective modeling of periodic heterogeneous beams. I. Asymptotic expansion method

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Abstract

This paper is concerned with the elastostatic behavior of heterogeneous beams with a cross-section and elastic moduli varying periodically along the beam axis. By using the two-scale asymptotic expansion method, the interior solution is formally derived up to an arbitrary desired order. In particular, this method is shown to constitute a systematic way of improving Bernoulli's theory by including higher-order terms, without any assumption, in contrast to Timoshenko's theory or other refined beam models. Moreover, the incompatibility between the interior asymptotic expansions and the real boundary conditions is emphasized, and the necessity of a specific treatment of edge effects is thus underlined. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Heterogeneous structures consisting of elements arranged periodically are widely used in civil engineering and industry. Using standard numerical methods (such as finite elements) to predict the overall behavior of these structures leads to heavy computations. However, when the size of the heterogeneity is small compared with the macroscopic dimension of the structure, the latter can be regarded as a homogeneous continuous medium. Thus, the method of homogenization can be applied.

The study presented here concerns the homogenization of structures having one large global dimension in comparison with the others, and a periodic heterogeneity only in this direction. One can for example think about repetitive lattice structures or any other periodic structure displaying overall beamlike behavior. Such structures possess two small parameters: e, which measures the ratio of the width of the cross-section to the total length L of the structure, and ε, which is the ratio of the length of the heterogeneity to the length L.

The method of homogenization consists in letting these two small parameters tend to zero starting from the three-dimensional (3D) elasticity problem. Hence several methods exist, depending on the order in

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which one realizes these two limits. The commutativity of the limiting processes has been studied from a theoretical point of view by Geymonat et al. (1987). The authors established that for a beam with a variable cross-section (transversal heterogeneity), different one-dimensional (1D) homogenized models are obtained when letting $e \rightarrow 0$ then $\varepsilon \rightarrow 0$, or the inverse. More precisely, the use of convergence theorems leads in both cases to a limit behavior corresponding to Bernoulli's model, but associated with different effective stiffnesses. The method consisting in letting first e tend to zero and then ε amounts to study a Bernoulli's beam with rapidly varying properties (Cioranescu and Saint Jean Paulin, 1999; Miller, 1994). Following the other method, we take the limit with respect to ε first (which corresponds to averaging the effect of cross-section and material variations) and afterwards the limit with respect to e (which consists in applying Bernoulli's theory to the resulting beam).

Another way of homogenization is to assume that the two small parameters simultaneously become vanishingly small. It leads to apply the method of the asymptotic expansion with only one small parameter. This approach has been initiated in Caillerie (1984) for periodic plates, and extended to the case of periodic beams in Kolpakov (1991) and Kalamkarov and Kolpakov (1997). At the first order, this method leads to a generalization of Euler–Bernoulli–Navier's model. The way of obtaining this limit behavior is widely explained in Kolpakov (1991), where convergence results are also established.

Therefore, three methods are available to homogenize the structure, and the question of defining their respective range of applicability naturally arises. When the limit processes are carried out successively, the method is a priori valid only if the parameter tending first to zero is much smaller than the other one. On the other hand, the method consisting in letting both parameters simultaneously tend to zero is a priori appropriate if e and ε are of the same order of magnitude, i.e. if the basic cell is neither very long and thin, nor very short and fat. However, the application of the latter method to different examples shows that its domain of validity can be enlarged (Buannic and Cartraud, 1999). In that reference, a periodic lattice structure is studied. The equivalent characteristics are identified from a classical study of a beam made of a large number of basic cells, and compared to those obtained from the two homogenization methods: method 1 ($e \rightarrow 0$ then $\varepsilon \rightarrow 0$) and method 2 ($e \simeq \varepsilon \rightarrow 0$). It turns out that the method 2 gives very accurate results whatever the value of the ratio e/ε, while the method 1 is valid if $e \ll \varepsilon$. Similar results have been obtained in the case of honeycomb plates (Bourgeois, 1997). This is in the same line as the conclusion drawn in Lewiński (1991b), in which the author claims that the only restrictions to the method 2 are $e \ll 1$ and $\varepsilon \ll 1$. The latter method will therefore be applied here.

The present paper aims at deriving the successive terms of the interior asymptotic expansions for periodic heterogeneous beams. As already mentioned, the first order terms correspond to Bernoulli's model. Consequently, the latter give a good approximation of the 3D behavior only if $\varepsilon \ll 1$ or if the applied loading does not involve any transverse shearing force within the structure. But in practice, ε is never infinitely small, and it may be necessary to characterize the higher-order terms of the expansions, which is the purpose of this paper. The expression of these terms is well known in the case of homogeneous isotropic elastic beams from Cimetière et al. (1988) in the nonlinear case, Rigolot (1976), Fan and Widera (1990) or Trabucho and Viaño (1996) in the linear case for an arbitrary cross-section, and Duva and Simmonds (1991) for a narrow rectangular cross-section treated in plane stress analysis. The case of transversely nonhomogeneous isotropic rods is also treated in Trabucho and Viaño (1996). We extend here these works to a periodic heterogeneous beam, with arbitrary variable cross-section, and within the framework of anisotropic elasticity.

Section 2 contains the formulation of the initial 3D elasticity problem and the definition of the notations. In Section 3, the asymptotic expansion method will be presented. It leads to a sequence of microscopic cellular problems (Section 4) as well as successive macroscopic 1D models (Section 5).

In this part, most of the attention is focused on the outer expansion of the beam equations. The treatment of end effects and the derivation of the boundary conditions will be given in Part II of this paper.

Throughout this paper, Latin indices take values in the set $\{1, 2, 3\}$ while Greek indices in $\{1, 2\}$. We also use the Einstein summation convention on repeated indices. Moreover, the partial derivatives $\partial/\partial z_3$, $\partial^2/\partial z_3^2$ and $\partial^3/\partial z_3^3$ will be denoted ∂_3, ∂_{33} and ∂_{333}.

2. The initial three-dimensional problem

The 3D slender structure Ω^ε considered herein is formed by periodic repetition of the periodicity cell Y^ε over the $\mathbf{e_3}$ direction (see Fig. 1).

Any kind of heterogeneity, geometrical or material, can be studied, and the structure is not assumed to present any particular symmetry (material or geometrical) with respect to the middle axis $x_1 = x_2 = 0$.

The periodicity cell Y^ε is defined by (see Fig. 2):

$$Y^\varepsilon = \left\{ \underset{\sim}{\mathbf{x}} = (x_i)/l_{1-}^\varepsilon(x_2, x_3) < x_1 < l_{1+}^\varepsilon(x_2, x_3);\ l_{2-}^\varepsilon(x_1, x_3) < x_2 < l_{2+}^\varepsilon(x_1, x_3);\ -\frac{l_3^\varepsilon}{2} < x_3 < \frac{l_3^\varepsilon}{2} \right\} \tag{1}$$

where the functions $l_{\alpha\pm}^\varepsilon$ are assumed to be periodic in x_3 with period l_3^ε. Let $Y^{*\varepsilon}$ be the solid part of the cell with boundary $\partial Y^{*\varepsilon}$ (see Fig. 2) such that $\partial Y^{*\varepsilon} = \partial Y_a^\varepsilon \cup \partial Y_b^\varepsilon \cup \partial Y_c^\varepsilon$ with ∂Y_a^ε the plane surfaces perpendicular to the $\mathbf{e_3}$ direction, ∂Y_b^ε the lateral outer boundary of the cell and ∂Y_c^ε its inner boundary (cell holes).

The elastic moduli of the beam, $a_{ijkl}^\varepsilon(\underset{\sim}{\mathbf{x}})$, are periodic in x_3 with period l_3^ε, and satisfy the following classical relations:

(i) $a_{ijkl}^\varepsilon(\underset{\sim}{\mathbf{x}}) = a_{jikl}^\varepsilon(\underset{\sim}{\mathbf{x}}) = a_{klij}^\varepsilon(\underset{\sim}{\mathbf{x}}), \quad \underset{\sim}{\mathbf{x}} \in \Omega^\varepsilon$

(ii) $\exists\, m > 0 \text{ such that } \forall \boldsymbol{\tau}/\tau_{ij} = \tau_{ji}, \quad m\tau_{ij}\tau_{ij} \leqslant a_{ijkl}^\varepsilon(\underset{\sim}{\mathbf{x}})\tau_{ij}\tau_{kl}$ (2)

(iii) $\exists\, M \text{ such that } M = \sup a_{ijkl}^\varepsilon(\underset{\sim}{\mathbf{x}}), \quad \underset{\sim}{\mathbf{x}} \in \Omega^\varepsilon$

The boundary of the domain Ω^ε is defined by $\partial\Omega^\varepsilon = S_0^\varepsilon \cup S_L^\varepsilon \cup \Gamma_b^\varepsilon \cup \Gamma_c^\varepsilon$, with S_0^ε and S_L^ε the two end sections of the beam and Γ_b^ε, Γ_c^ε obtained from the periodic repetition of ∂Y_b^ε, ∂Y_c^ε respectively (see Fig. 1).
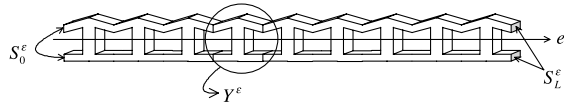

Fig. 1. 3D slender periodic structure Ω^ε.
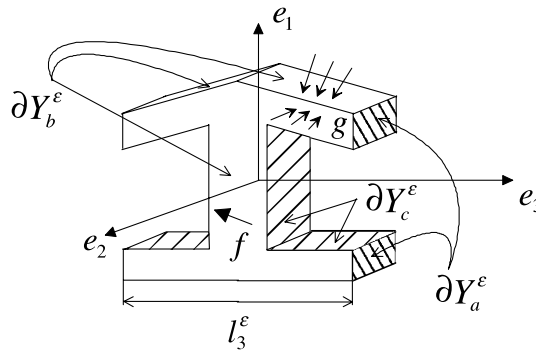

Fig. 2. Periodicity cell Y^ε.

The beam is considered to be under body forces $\mathbf{f}^{\varepsilon}$ and tractions $\mathbf{g}^{\varepsilon}$ on the outer boundary Γ_b^{ε} (see Fig. 2). The holes boundary Γ_c^{ε} is supposed to be free of traction. The left end S_0^{ε} is clamped and stress data $\bar{\sigma}_{3i}^{\varepsilon}(x_1, x_2)$ are prescribed on the right end S_L^{ε}.

The static problem P^{ε} of linear elasticity consists in finding the fields $\boldsymbol{\sigma}^{\varepsilon}$, $\mathbf{e}^{\varepsilon}$ and $\mathbf{u}^{\varepsilon}$, such that:

$$
\begin{cases}
\mathbf{div_x}\, \boldsymbol{\sigma}^{\varepsilon} = -\mathbf{f}^{\varepsilon} \\
\boldsymbol{\sigma}^{\varepsilon} = \mathbf{a}^{\varepsilon}(\mathbf{x}) : \mathbf{e}^{\varepsilon}(\mathbf{u}^{\varepsilon}) \\
\mathbf{e}^{\varepsilon}(\mathbf{u}^{\varepsilon}) = \mathbf{grad_x^s}(\mathbf{u}^{\varepsilon}) \\
\boldsymbol{\sigma}^{\varepsilon} \cdot \mathbf{n} = \mathbf{g}^{\varepsilon} \quad \text{on } \Gamma_b^{\varepsilon} \\
\boldsymbol{\sigma}^{\varepsilon} \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_c^{\varepsilon} \\
\boldsymbol{\sigma}^{\varepsilon} \cdot \mathbf{e_3} = \bar{\sigma}_{3i}^{\varepsilon}(x_1, x_2) \cdot \mathbf{e_i} \quad \text{on } S_L^{\varepsilon} \\
\mathbf{u}^{\varepsilon} = \mathbf{0} \quad \text{on } S_0^{\varepsilon}
\end{cases}
\tag{3}
$$

where $\mathbf{grad_x^s}$ and $\mathbf{div_x}$ correspond respectively to the symmetric strain and divergence operators, with respect to the spatial coordinate $\mathbf{x}$. The vectors $\mathbf{n}$ in Eq. (3) (third and fourth equations) and $\mathbf{e_3}$ in Eq. (3) (fifth equation) denote the outer normal of the corresponding boundary. The superscript ε in the formulation of P^{ε} indicates that the solutions depend on the values of the two small parameters of the structure, e and ε, which are assumed to be equal, as explained in Section 1.

A unique solution $\boldsymbol{\sigma}^{\varepsilon}$, $\mathbf{e}^{\varepsilon}$, $\mathbf{u}^{\varepsilon}$ exists for the problem (3) under conditions (2) and assuming that the functions $\mathbf{f}^{\varepsilon}$, $\mathbf{g}^{\varepsilon}$ and $\bar{\sigma}_{i3}^{\varepsilon}(x_1, x_2)$ are sufficiently smooth, and the boundary $\partial\Omega^{\varepsilon}$ regular.

3. The asymptotic expansion method

The first step of the method consists in defining a problem equivalent to the problem (3), but now posed on a fixed domain which does not depend on the small parameter ε.

To this end, we apply the technique of Caillerie (1984) and Kolpakov (1991), and so introduce the following changes of variables, to take into account successively the slenderness of the beam cross-section and the smallness of the beam heterogeneity:

$$
\begin{aligned}
&(z_1, z_2, z_3) = \left(\frac{x_1}{e}, \frac{x_2}{e}, x_3 \right), \\
&(y_1, y_2, y_3) = \left(z_1, z_2, \frac{z_3}{\varepsilon} \right) = \frac{1}{\varepsilon}(x_1, x_2, x_3) \quad \text{since } e = \varepsilon
\end{aligned}
\tag{4}
$$

Consequently, z_3 represents the slow or large scale or *macroscopic* variable of the problem and $\mathbf{y} = \mathbf{x}/\varepsilon$ the fast or small scale or *microscopic* one.

According to this change of variable, we associate the new strain and divergence operators in the following manner:

$$
\begin{cases}
\mathbf{grad_x^s}\,. = \mathbf{grad_{z_3}^s}\,. + \frac{1}{\varepsilon}\mathbf{grad_y^s}\,. \\
\mathbf{div_x}\,. = \mathbf{div_{z_3}}\,. + \frac{1}{\varepsilon}\mathbf{div_y}\,.
\end{cases}
\tag{5}
$$

where $\mathbf{grad_{z_3}^s}$ and $\mathbf{div_{z_3}}$ correspond to partial differentiations with respect to the only variable z_3, while $\mathbf{grad_y^s}$ and $\mathbf{div_y}$ are the differential operators with regard to the three microscopic variables y_i.

As a second step, it is necessary to presuppose the order of magnitude of the loadings which are applied to the structure. Especially, we set:

$$f_3^\varepsilon(x_1, x_2, x_3) = \varepsilon^1 \cdot f_3(z_3, y_1, y_2) \quad g_3^\varepsilon(x_1, x_2, x_3) = \varepsilon^2 \cdot g_3(z_3, y_1, y_2)$$
$$f_\alpha^\varepsilon(x_1, x_2, x_3) = \varepsilon^2 \cdot f_\alpha(z_3, y_1, y_2) \quad g_\alpha^\varepsilon(x_1, x_2, x_3) = \varepsilon^3 \cdot g_\alpha(z_3, y_1, y_2) \tag{6}$$
$$\bar{\sigma}_{33}^\varepsilon(x_1, x_2) = \varepsilon^1 \cdot \bar{\sigma}_{33}(y_1, y_2) \quad \bar{\sigma}_{\alpha 3}^\varepsilon(x_1, x_2) = \varepsilon^2 \cdot \bar{\sigma}_{\alpha 3}(y_1, y_2)$$

Furthermore, the elasticity moduli a_{ijkl}^ε are assumed to be independent of ε, so we have:

$$\mathbf{a}^\varepsilon(\underset{\sim}{\mathbf{x}}) = \mathbf{a}(\underset{\sim}{\mathbf{y}}) \tag{7}$$

Remark 3.1. The homogenized limit 1D model depend on the orders of magnitude of the applied loadings with respect to ε, for example Karwowski (1990) for the theory of asymptotic modeling of rods, or Caillerie (1980) and Millet (1997) for the case of plates. The assumptions (6) and (7) are usually made in order that the limit behavior ($\varepsilon \to 0$) of the 3D slender structure is that of a beam (Trabucho and Viaño, 1996; Cimetière et al., 1988).

Third, following a standard technique, the solution $\underset{\sim}{\mathbf{u}}^\varepsilon$ of the (P^ε) problem is sought in the form used in Kolpakov (1991):

$$\underset{\sim}{\mathbf{u}}^\varepsilon(\underset{\sim}{\mathbf{x}}) = \hat{u}_\alpha^0(z_3)\underset{\sim}{\mathbf{e}_\alpha} + \varepsilon\underset{\sim}{\mathbf{u}}^1(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon^2\underset{\sim}{\mathbf{u}}^2(z_3, \underset{\sim}{\mathbf{y}}) + \cdots \tag{8}$$

where every function $\underset{\sim}{\mathbf{u}}^k(z_3, y_i)$ is periodic in the variable y_3 with period l_3 ($l_3 = l_3^\varepsilon/\varepsilon$), which will be denoted y_3-periodic in the following.

Consequently, using Eq. (5) (first equation) and the constitutive relations, the strains and stresses expansions are given by:

$$\mathbf{e}^\varepsilon(\underset{\sim}{\mathbf{x}}) = \mathbf{e}^0(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon\mathbf{e}^1(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon^2\mathbf{e}^2(z_3, \underset{\sim}{\mathbf{y}}) + \cdots$$
$$\boldsymbol{\sigma}^\varepsilon(\underset{\sim}{\mathbf{x}}) = \boldsymbol{\sigma}^0(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon\boldsymbol{\sigma}^1(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon^2\boldsymbol{\sigma}^2(z_3, \underset{\sim}{\mathbf{y}}) + \cdots \tag{9}$$

Remark 3.2. The form of the first term of the expansion (8), which is composed of only the deflections $\hat{u}_\alpha^0(z_3)$, is not an assumption, in the sense that applying the asymptotic method with the relations (6) leads to that expression of $\mathbf{u}^\varepsilon$ (Trabucho and Viaño, 1996). A similar result has been established in the case of periodic plates (Caillerie, 1984), where the first term of the expansion is found to be reduced to the deflection of the middle plane (i.e. $u_3^0(z_\alpha)\mathbf{e_3}$) if appropriate magnitude order assumptions are made on the applied loadings.

Because the beam asymptotic model obtained under assumptions (6) satisfies the relation (8), it is usual to scale the displacement components (Trabucho and Viaño, 1996; Cioranescu and Saint Jean Paulin, 1999). For example, in Trabucho and Viaño (1996), the authors associate with the displacement field $\underset{\sim}{\mathbf{u}}^\varepsilon(\underset{\sim}{\mathbf{x}})$ the scaled functions $\underset{\sim}{\mathbf{u}}(z_3, \underset{\sim}{\mathbf{y}})(\varepsilon)$ through the following scalings:

$$\begin{cases} u_\alpha^\varepsilon(\underset{\sim}{\mathbf{x}}) = \varepsilon^{-1} u_\alpha(z_3, \underset{\sim}{\mathbf{y}})(\varepsilon) \\ u_3^\varepsilon(\underset{\sim}{\mathbf{x}}) = u_3(z_3, \underset{\sim}{\mathbf{y}})(\varepsilon) \end{cases} \tag{10}$$

and these scaled functions are then assumed to have the asymptotic expansion:

$$u_i(z_3, \underset{\sim}{\mathbf{y}})(\varepsilon) = u_i^0(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon u_i^1(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon^2 u_i^2(z_3, \underset{\sim}{\mathbf{y}}) + \cdots$$

The justification of the scalings (10) is purely mathematical and is related to convergence results of the asymptotic method as $\varepsilon \to 0$. However, this convergence aspect will not be treated at all in the present

paper. Thus, such scalings will not be used here, since all the results presented here remain *formal*, i.e. without any convergence study.

When introducing the relations (4)–(8) into the P^ε problem (3) and equating the terms of a same order with respect to ε, we replace the problem P^ε by a family of problems. The fields involved in the latter are functions of the two kinds of variables y_i and z_3, but no longer depend on the small parameter ε. So, when treating the z_3- and y_i-coordinates as independent, and considering the fields function of the only variable z_3 as given data, we can regard each of these problems as a *microscopic* problem, which is posed on the scaled period Y. In that sense, these successive problems are commonly named the cellular periodic (or basic cell) problems, and will be denoted herein P^k_{cell}, where the superscript k stands for the order of the current problem with respect to ε. As it will be seen in Section 4, the solution of these cellular problems enables us to determine the periodic parts of the expansions (8) and (9). Then, expressing the existence conditions of solutions for the P^k_{cell} problems, we obtain the formulation of homogenized 1D-macroscopic problems, denoted by P^k_{hom}, the solution of which gives the macroscopic (i.e. nonperiodic) parts of the fields (8) and (9). The formulation and solution of the P^k_{hom} problems are treated in Section 5.

4. The set of cellular problems P^k_{cell}

From the change of variable (4) and of operators (5), and inserting the asymptotic expansion (8) for the displacement field $\mathbf{u}^\varepsilon$ into the initial P^ε problem, one can derive an infinite set of cellular P^k_{cell} problems, with k starting from -1.

4.1. General formulation of the kth cellular problem P^k_{cell}

For an arbitrary power k of the small parameter ε, the P^k_{cell} problem is posed on the fixed period Y defined as:

$$Y = \left\{ \underset{\sim}{\mathbf{y}} = (y_i)/l_{1-}(y_2, y_3) < y_1 < l_{1+}(y_2, y_3); \ l_{2-}(y_1, y_3) < y_2 < l_{2+}(y_1, y_3); \ -\frac{l_3}{2} < y_3 < \frac{l_3}{2} \right\}$$

$$\text{with } l_{\alpha\pm}(y_\beta, y_3) = l^\varepsilon_{\alpha\pm}(\varepsilon y_\beta, \varepsilon y_3)/\varepsilon \ \text{ and } \ l_3 = l^\varepsilon_3/\varepsilon,$$

and with boundary ∂Y^* such that $\partial Y^* = \partial Y_a \cup \partial Y_b \cup \partial Y_c$. ∂Y_a, ∂Y_b, ∂Y_c denote the scaled lateral boundary surface obtained from ∂Y^ε_a, ∂Y^ε_b, ∂Y^ε_c respectively.

The P^k_{cell} problem consists in finding the fields $\boldsymbol{\sigma}^{k+1}$, $\mathbf{e}^{k+1}$ and $\underset{\sim}{\mathbf{u}}^{k+2}$ satisfying the following equations:

$$\begin{cases}
\mathbf{div}_{\mathbf{y}} \, \boldsymbol{\sigma}^{k+1} = -\underset{\sim}{\mathbf{f}}^k - \mathbf{div}_{z_3} \, \boldsymbol{\sigma}^k \\[4pt]
\boldsymbol{\sigma}^{k+1} = \mathbf{a}(\underset{\sim}{\mathbf{y}}) : \mathbf{e}^{k+1} \\[4pt]
\mathbf{e}^{k+1} = \mathbf{grad}^{\mathbf{s}}_{\mathbf{y}}(\underset{\sim}{\mathbf{u}}^{k+2}) + \mathbf{grad}^{\mathbf{s}}_{z_3}(\underset{\sim}{\mathbf{u}}^{k+1}) \\[4pt]
\boldsymbol{\sigma}^{k+1} \cdot \underset{\sim}{\mathbf{n}} = \underset{\sim}{\mathbf{g}}^{k+1} \quad \text{on } \partial Y_b \\[4pt]
\boldsymbol{\sigma}^{k+1} \cdot \underset{\sim}{\mathbf{n}} = \underset{\sim}{\mathbf{0}} \quad \text{on } \partial Y_c \\[4pt]
\sigma^{k+1}_{i3} \ \text{and} \ \underset{\sim}{\mathbf{u}}^{k+2} \ y_3\text{-periodic}
\end{cases} \tag{12}$$

with $k \geqslant -1$ and where the negative powers of $\boldsymbol{\sigma}^k$ and $\mathbf{e}^k$ vanish. $\underset{\sim}{\mathbf{f}}^k$ and $\underset{\sim}{\mathbf{g}}^k$ correspond respectively to the body and surface densities of forces which occur at the order ε^k. We recall that these forces are assumed to obey the relations (6), so that $f^k_3 = g^{k+1}_3 = 0$ if $k \neq 1$ and that $f^k_\alpha = g^{k+1}_\alpha = 0$ if $k \neq 2$. Finally, the periodic

boundary conditions (12) (sixth equation) result from the structure periodicity, see Eq. (8), and due to the opposite values of $\boldsymbol{\sigma}^{k+1} \cdot \underset{\sim}{\mathbf{n}}$ on opposite sides of ∂Y_a where $\underset{\sim}{\mathbf{n}} = \pm \mathbf{e_3}$.

Remark 4.1. When solving the cellular problem P_{cell}^k at order k, we consider that the preceding P_{cell}^{k-1} problem has already been solved and thus that the fields $\boldsymbol{\sigma}^k$ and $\mathbf{u}^{k+1}$ have been determined. Consequently, the parameters $\mathbf{div}_{z_3} \boldsymbol{\sigma}^k$ and $\mathbf{grad}^s_{z_3}(\mathbf{u}^{k+1})$ constitute macroscopic *given* fields for the current problem P_{cell}^k: the first one can be regarded as a fictive volume force and the second as an initial strain state in the period Y.

Let us introduce $W(Y) = \{\boldsymbol{\psi} \in [H^1(Y)]^3, y_3\text{-periodic}\}$. The P_{cell}^k problem is equivalent to: find the displacement field $\mathbf{u}^{k+2}$ belonging to $W(Y)$ such that

$$\forall \underset{\sim}{\boldsymbol{\psi}} \in W(Y), \quad \int_{Y^*} \boldsymbol{\sigma}^{k+1} : \mathbf{grad}^s_{\mathbf{y}}(\underset{\sim}{\boldsymbol{\psi}}) \, \mathrm{d}Y = \int_{Y^*} (\mathbf{div}_{z_3} \boldsymbol{\sigma}^k + \underset{\sim}{\mathbf{f}}^k) \cdot \underset{\sim}{\boldsymbol{\psi}} \, \mathrm{d}Y + \int_{\partial Y_b} \mathbf{g}^{k+1} \cdot \underset{\sim}{\boldsymbol{\psi}} \, \mathrm{d}\Gamma \tag{13}$$

where the stress field $\boldsymbol{\sigma}^{k+1}$ is related to the displacement field $\mathbf{u}^{k+2}$ following Eq. (12) (second equation) and where $\mathrm{d}Y = \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}y_3$. According to the variational form Eq. (13), it is easy to show that the P_{cell}^k problem possesses a solution provided that the data $\mathbf{div}_{z_3} \boldsymbol{\sigma}^k$, $\mathbf{f}^k$, $\mathbf{g}^{k+1}$ verify the following relation:

$$\forall \underset{\sim}{\mathbf{v}} \in \mathfrak{R}, \quad \int_{Y^*} (\mathbf{div}_{z_3} \boldsymbol{\sigma}^k + \underset{\sim}{\mathbf{f}}^k) \cdot \underset{\sim}{\mathbf{v}} \, \mathrm{d}Y + \int_{\partial Y_b} \mathbf{g}^{k+1} \cdot \underset{\sim}{\mathbf{v}} \, \mathrm{d}\Gamma = 0 \tag{14}$$

where $\mathfrak{R}$ corresponds to the set of the y_3-periodic rigid body motions for the period Y, and is given by:

$$\mathfrak{R} = \{\underset{\sim}{\mathbf{v}}(z_3, \underset{\sim}{\mathbf{y}}) / \underset{\sim}{\mathbf{v}} = \hat{v}_i(z_3) \cdot \mathbf{e_i} + \varphi(z_3)[y_1 \cdot \mathbf{e_2} - y_2 \cdot \mathbf{e_1}]\} \tag{15}$$

Under the necessary condition (14), the solutions $\boldsymbol{\sigma}^{k+1}$, $\mathbf{e}^{k+1}$ and $\mathbf{u}^{k+2}$ (determined up to an element of $\mathfrak{R}$) exist and can be linearly expressed with respect to these data. The compatibility condition (14) will enable us to formulate the macroscopic problems, as we shall see in Section 5.

In the next sections, we give the solution of the cellular problems which leads to the determination of the microscopic parts of the displacement field $\underset{\sim}{\mathbf{u}}^{\varepsilon}$ and consequently to a formal expression of the latter.

4.2. Solution of the cellular problem P_{cell}^{-1}

The first cellular problem occurs for $k = -1$. Since we have assumed that no force $\underset{\sim}{\mathbf{f}}^{-1}$ or $\mathbf{g}^0$ is applied at this order, it can be written as follows:

$$\begin{cases} \mathbf{div_y} \boldsymbol{\sigma}^0 = \underset{\sim}{\mathbf{0}} \\ \boldsymbol{\sigma}^0 = \mathbf{a}(\underset{\sim}{\mathbf{y}}) : \mathbf{e}^0 \\ \mathbf{e}^0 = \mathbf{grad}^s_{\mathbf{y}}(\mathbf{u}^1) + \mathbf{grad}^s_{z_3}(\underset{\sim}{\mathbf{u}}^0) \\ \boldsymbol{\sigma}^0 \cdot \underset{\sim}{\mathbf{n}} = \underset{\sim}{\mathbf{0}} \quad \text{on } \partial Y_b \cup \partial Y_c \\ \sigma^0_{i3} \text{ and } \underset{\sim}{\mathbf{u}}^1 \ y_3\text{-periodic} \end{cases} \tag{16}$$

The only data of the problem are thus contained in the tensor $\mathbf{grad}^s_{z_3}(\underset{\sim}{\mathbf{u}}^0)$ and, according to the form of the field $\underset{\sim}{\mathbf{u}}^0$, we have:

$$\mathbf{grad}^s_{z_3}(\underset{\sim}{\mathbf{u}}^0) = \begin{bmatrix} 0 & 0 & \frac{1}{2}\partial_3 \hat{u}^0_1(z_3) \\ & 0 & \frac{1}{2}\partial_3 \hat{u}^0_2(z_3) \\ \text{sym} & & 0 \end{bmatrix} \tag{17}$$

where $\partial_3 \hat{u}_\alpha^0(z_3)$ are the two macroscopic data of the P_{cell}^{-1} problem and where sym stands for the symmetric part of the matrix.

The compatibility condition (14) is satisfied identically for the problem (16), ensuring thus the existence of the solution. Furthermore, one can easily establish that this problem possesses a direct solution which is:

$$\underset{\sim}{\mathbf{u}}^1{}_{\text{part}} = -y_\alpha \cdot \partial_3 \hat{u}_\alpha^0(z_3) \cdot \mathbf{e_3} \quad \text{and} \quad \boldsymbol{\sigma}^0 = \mathbf{e}^0 = 0 \tag{18}$$

In that sense, the two data $\partial_3 \hat{u}_\alpha^0(z_3)$ do not constitute effective data, since the associated solution corresponds to a zero deformation state (Sanchez-Hubert and Sanchez-Palencia, 1992).

The displacement field given in Eq. (18) is obtained up to an element of $\mathfrak{R}$, so the complete solution of the P_{cell}^{-1} problem has to be written:

$$\underset{\sim}{\mathbf{u}}^1 = \hat{u}_i^1(z_3) \cdot \mathbf{e_i} + \varphi^1(z_3)[y_1 \cdot \mathbf{e_2} - y_2 \cdot \mathbf{e_1}] - y_\alpha \cdot \partial_3 \hat{u}_\alpha^0(z_3) \cdot \mathbf{e_3} \equiv \underset{\sim}{\acute{\mathbf{u}}}^1(z_3, \underset{\sim}{\mathbf{y}}) \tag{19}$$

4.3. Solution of the zeroth order cellular problem P_{cell}^0

Since $\boldsymbol{\sigma}^0 = 0$, the P_{cell}^0 problem consists in finding the fields $\boldsymbol{\sigma}^1$, $\mathbf{e}^1$ and $\underset{\sim}{\mathbf{u}}^2$ which satisfy:

$$\begin{cases} \underset{\sim}{\mathbf{div_y}}\, \boldsymbol{\sigma}^1 = \underset{\sim}{\mathbf{0}} \\ \boldsymbol{\sigma}^1 = \mathbf{a}(\underset{\sim}{\mathbf{y}}) : \mathbf{e}^1 \\ \mathbf{e}^1 = \mathbf{grad_y^s}(\underset{\sim}{\mathbf{u}}^2) + \mathbf{grad_{z_3}^s}(\underset{\sim}{\mathbf{u}}^1) \\ \boldsymbol{\sigma}^1 \cdot \underset{\sim}{\mathbf{n}} = \underset{\sim}{\mathbf{0}} \quad \text{on } \partial Y_b \cup \partial Y_c \\ \sigma_{i3}^1 \text{ and } \underset{\sim}{\mathbf{u}}^2 \ y_3\text{-periodic} \end{cases} \tag{20}$$

As with the preceding problem, the compatibility condition (14) is satisfied identically for the problem (20). According to the expression (19) of $\underset{\sim}{\mathbf{u}}^1$ obtained at the preceding order, the data of the zeroth order cellular problem can be written as follows:

$$\mathbf{grad_{z_3}^s}(\underset{\sim}{\mathbf{u}}^1) \equiv \begin{bmatrix} 0 & 0 & \frac{1}{2}(\partial_3 \hat{u}_1^1(z_3) - y_2 \partial_3 \varphi^1(z_3)) \\ & 0 & \frac{1}{2}(\partial_3 \hat{u}_2^1(z_3) + y_1 \partial_3 \varphi^1(z_3)) \\ \text{sym} & & (\partial_3 \hat{u}_3^1(z_3) - y_\alpha \partial_{33} \hat{u}_\alpha^0(z_3)) \end{bmatrix} \tag{21}$$

The two data $\partial_3 \hat{u}_\alpha^1(z_3)$ will provide a direct solution $\underset{\sim}{\mathbf{u}}^2{}_{\text{part}}$ similar to expression (18). The four other data which are contained in $\mathbf{grad_{z_3}^s}(\underset{\sim}{\mathbf{u}}^1)$, namely $\partial_3 \hat{u}_3^1(z_3)$, $\partial_{33} \hat{u}_\alpha^0(z_3)$, $\partial_3 \varphi^1(z_3)$, correspond respectively to a macroscopic extension, two macroscopic curvatures and a macroscopic torsion rotation. Due to the linearity of the problem (20), the displacement field $\underset{\sim}{\mathbf{u}}^2$ can be expressed as a linear function of these four effective data. Adding the direct solution $\underset{\sim}{\mathbf{u}}^2{}_{\text{part}}$ provided by the two other data $\partial_3 \hat{u}_\alpha^1(z_3)$ as well as the rigid motion, the complete displacement field at the second order assumes the following form:

$$\underset{\sim}{\mathbf{u}}^2 = \underset{\sim}{\acute{\mathbf{u}}}^2 + \underset{\sim}{\boldsymbol{\chi}}^{1E}(\underset{\sim}{\mathbf{y}}) \cdot \partial_3 \hat{u}_3^1(z_3) + \underset{\sim}{\boldsymbol{\chi}}^{1C_\alpha}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{33} \hat{u}_\alpha^0(z_3) + \underset{\sim}{\boldsymbol{\chi}}^{1T}(\underset{\sim}{\mathbf{y}}) \cdot \partial_3 \varphi^1(z_3) \tag{22}$$

where $\underset{\sim}{\acute{\mathbf{u}}}^2(z_3, \underset{\sim}{\mathbf{y}}) = \hat{u}_i^2(z_3) \cdot \mathbf{e_i} + \varphi^2(z_3)[y_1 \cdot \mathbf{e_2} - y_2 \cdot \mathbf{e_1}] - y_\alpha \cdot \partial_3 \hat{u}_\alpha^1(z_3) \cdot \mathbf{e_3}$. For later consistency of notations, we introduce the four-components vector $\underset{\sim}{\acute{\mathbf{e}}}^1(z_3)$ and the 3×4 matrix $\underset{\sim}{\boldsymbol{\chi}}^1(\underset{\sim}{\mathbf{y}})$ so that we have:

$$\underset{\sim}{\mathbf{u}}^2 = \underset{\sim}{\acute{\mathbf{u}}}^2(z_3, \underset{\sim}{\mathbf{y}}) + \underset{\sim}{\boldsymbol{\chi}}^1(\underset{\sim}{\mathbf{y}}) \cdot \underset{\sim}{\acute{\mathbf{e}}}^1(z_3) \tag{23}$$

with

$$\dot{\underset{\sim}{\mathbf{e}}}^1(z_3) =^{\mathrm{t}} \{\partial_3 \hat{u}_3^1(z_3), \partial_{33} \hat{u}_1^0(z_3), \partial_{33} \hat{u}_2^0(z_3), \partial_3 \varphi^1(z_3)\} \tag{24}$$

$$\underset{\sim}{\boldsymbol{\chi}}^1(\mathbf{y}) = [\underset{\sim}{\boldsymbol{\chi}}^{1E}(\mathbf{y}), \underset{\sim}{\boldsymbol{\chi}}^{1C_1}(\mathbf{y}), \underset{\sim}{\boldsymbol{\chi}}^{1C_2}(\mathbf{y}), \underset{\sim}{\boldsymbol{\chi}}^{1T}(\mathbf{y})] \tag{25}$$

In Eq. (24), the four effective data have been grouped in the vector $\dot{\underset{\sim}{\mathbf{e}}}^1(z_3)$, with the result that the latter represents the first order macroscopic strain vector.

Remark 4.2. The problem (20) does not have an analytical form solution for the unknown $\underset{\sim}{\boldsymbol{\chi}}^1(\mathbf{y})$ in general, except in the case of homogeneous rods, see for example Trabucho and Viaño (1996) for an isotropic material.

In the same manner as the displacement field, the stress field $\boldsymbol{\sigma}^1$ solution of P_{cell}^0 has a linear expression with regard to the data:

$$\boldsymbol{\sigma}^1 = \underset{\sim}{\boldsymbol{\tau}}^{1E}(\mathbf{y}) \cdot \partial_3 \hat{u}_3^1(z_3) + \underset{\sim}{\boldsymbol{\tau}}^{1C_\alpha}(\mathbf{y}) \cdot \partial_{33} \hat{u}_\alpha^0(z_3) + \underset{\sim}{\boldsymbol{\tau}}^{1T}(\mathbf{y}) \cdot \partial_3 \varphi^1(z_3) \tag{26}$$

with

$$\begin{cases} \tau_{ij}^{1E} = a_{ij33} + a_{ijkh}\partial_{y_h}\chi_k^{1E} \\ \tau_{ij}^{1C_\alpha} = -y_\alpha a_{ij33} + a_{ijkh}\partial_{y_h}\chi_k^{1C_\alpha} \\ \tau_{ij}^{1T} = -y_2 a_{ij13} + y_1 a_{ij23} + a_{ijkh}\partial_{y_h}\chi_k^{1T} \end{cases} \tag{27}$$

which will be formally denoted as:

$$\boldsymbol{\sigma}^1 = \underset{\sim}{\boldsymbol{\tau}}^1(\mathbf{y}) \cdot \dot{\underset{\sim}{\mathbf{e}}}^1(z_3) \tag{28}$$

where $\underset{\sim}{\boldsymbol{\tau}}^1(\mathbf{y})$ corresponds to the regrouping of the four elementary stress tensors $\boldsymbol{\tau}^{1E}, \boldsymbol{\tau}^{1C_\alpha}, \boldsymbol{\tau}^{1T}$ so that:

$$\sigma_{ij}^1 = \tau_{ij1}^1(\mathbf{y}) \cdot \partial_3 \hat{u}_3^1(z_3) + \tau_{ij2}^1(\mathbf{y}) \cdot \partial_{33}\hat{u}_1^0(z_3) + \tau_{ij3}^1(\mathbf{y}) \cdot \partial_{33}\hat{u}_2^0(z_3) + \tau_{ij4}^1(\mathbf{y}) \cdot \partial_3 \varphi^1(z_3) = \tau_{ijm}^1 \dot{e}_m^1, \quad m \in [1, 4] \tag{29}$$

with

$$\tau_{ij1}^1 = \tau_{ij}^{1E}; \quad \tau_{ij2}^1 = \tau_{ij}^{1C_1}; \quad \tau_{ij3}^1 = \tau_{ij}^{1C_2}; \quad \tau_{ij4}^1 = \tau_{ij}^{1T} \tag{30}$$

4.4. Solution of the first order cellular problem P_{cell}^1

It follows from Eqs. (6), (22) and (26) that the P_{cell}^1 problem comprises the following relations:

$$\begin{cases} \underset{\sim}{\mathbf{div}}_{\mathbf{y}} \boldsymbol{\sigma}^2 = -\underset{\sim}{\mathbf{div}}_{z_3} \boldsymbol{\sigma}^1 - f_3 \cdot \mathbf{e_3} \\ \boldsymbol{\sigma}^2 = \mathbf{a}(\mathbf{y}) : \mathbf{e}^2 \\ \mathbf{e}^2 = \mathbf{grad}_{\mathbf{y}}^{\mathrm{s}}(\mathbf{u}^3) + \mathbf{grad}_{z_3}^{\mathrm{s}}(\mathbf{u}^2) \\ \boldsymbol{\sigma}^2 \cdot \underline{\mathbf{n}} = g_3 \mathbf{e_3} \quad \text{on } \partial Y_b \\ \boldsymbol{\sigma}^2 \cdot \underline{\mathbf{n}} = \underset{\sim}{\mathbf{0}} \quad \text{on } \partial Y_c \\ \sigma_{i3}^2 \text{ and } \mathbf{u}^3 \ y_3\text{-periodic} \end{cases} \tag{31}$$

with

$$\mathbf{div}_{\underset{\sim}{z_3}} \boldsymbol{\sigma}^1 = (\tau_{i3}^{1E}(\mathbf{y}) \cdot \partial_{33}\hat{u}_3^1(z_3) + \tau_{i3}^{1C_\alpha}(\mathbf{y}) \cdot \partial_{333}\hat{u}_\alpha^0(z_3) + \tau_{i3}^{1T}(\mathbf{y}) \cdot \partial_{33}\varphi^1(z_3)) \cdot \mathbf{e}_{\underset{\sim}{i}} \tag{32}$$

and

$$\mathbf{grad}^{\mathbf{s}}_{\underset{\sim}{z_3}}(\mathbf{u}^2) = \begin{bmatrix} 0 & 0 & \frac{1}{2}(\partial_3\hat{u}_1^2(z_3) - y_2\partial_3\varphi^2(z_3)) \\ & 0 & \frac{1}{2}(\partial_3\hat{u}_2^2(z_3) + y_1\partial_3\varphi^2(z_3)) \\ \text{sym} & & (\partial_3\hat{u}_3^2(z_3) - y_\alpha\partial_{33}\hat{u}_\alpha^1(z_3)) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \frac{1}{2}(\chi_1^{1E}(\mathbf{y})\partial_{33}\hat{u}_3^1(z_3) + \chi_1^{1C_\alpha}(\mathbf{y})\partial_{333}\hat{u}_\alpha^0(z_3) + \chi_1^{1T}(\mathbf{y})\partial_{33}\varphi^1(z_3)) \\ & 0 & \frac{1}{2}(\chi_2^{1E}(\mathbf{y})\partial_{33}\hat{u}_3^1(z_3) + \chi_2^{1C_\alpha}(\mathbf{y})\partial_{333}\hat{u}_\alpha^0(z_3) + \chi_2^{1T}(\mathbf{y})\partial_{33}\varphi^1(z_3)) \\ \text{sym} & & (\chi_3^{1E}(\mathbf{y})\partial_{33}\hat{u}_3^1(z_3) + \chi_3^{1C_\alpha}(\mathbf{y})\partial_{333}\hat{u}_\alpha^0(z_3) + \chi_3^{1T}(\mathbf{y})\partial_{33}\varphi^1(z_3)) \end{bmatrix} \tag{33}$$

This problem admits a solution up to an element of $\mathfrak{R}$ if and only if the data $\mathbf{div}_{\underset{\sim}{z_3}}\boldsymbol{\sigma}^1$, $(0,0,f_3)$ and $(0,0,g_3)$ satisfy the relation (14). From Eq. (32), $\mathbf{div}_{\underset{\sim}{z_3}}\boldsymbol{\sigma}^1$ can be expressed as a function of $\partial_3\mathbf{\acute{e}}^1$, which is the first gradient of the strains $\mathbf{\acute{e}}^1(z_3)$. Thus, the compatibility conditions (14) lead to a relation between $\partial_3\mathbf{\acute{e}}^1$ and f_3 and g_3, which enables us to express the fictive volume force $\mathbf{div}_{\underset{\sim}{z_3}}\boldsymbol{\sigma}^1$ in the form:

$$\mathbf{div}_{\underset{\sim}{z_3}}\boldsymbol{\sigma}^1 = \mathbf{h}_1(\partial_3\mathbf{\acute{e}}^1) + \mathbf{h}_2(f_3) + \mathbf{h}_3(g_3) \tag{34}$$

where $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$ are linear functions. The latter expression is such that if in the problem (31) only the data involving f_3 and g_3, i.e. the body forces $\mathbf{h}_2(f_3) + \mathbf{h}_3(g_3) + f_3\mathbf{e}_3$ and tractions $g_3\mathbf{e}_3$ on ∂Y_b, then a well-posed problem is found. In the same way, the problems involving the other data, i.e. the body forces $\mathbf{h}_1(\partial_3\mathbf{\acute{e}}^1)$ and the initial strain state $\mathbf{grad}^{\mathbf{s}}_{\underset{\sim}{z_3}}(\mathbf{\grave{u}})$, are also well posed. A more complete treatment of that question will be given later (in Section 5.1.5), once the compatibility relations of the problem (31) have been expressed. Let us study now the form of the solutions $\mathbf{u}^3$ and $\boldsymbol{\sigma}^2$ of the problem (14), which can be linearly expressed with respect to the data, in the same manner as at the preceding orders.

Firstly, the solution of the well posed elementary problem corresponding to the prescribed data f_3, g_3 only is denoted by $\mathbf{\acute{u}}^3_{\text{part}}$. The other data of the problem come from Eq. (33) and $\mathbf{h}_1(\partial_3\mathbf{\acute{e}}^1)$ in Eq. (34). In order to give the form of the solution with respect to these data, it must be noticed here that the first matrix on the right side of Eq. (33) is identical to the data matrix (21) of the preceding P^0_{cell} problem, except that the superscripts have increased by one. As a consequence, the set of these six data, namely $\partial_3\hat{u}_i^2(z_3)$, $\partial_{33}\hat{u}_\alpha^1(z_3)$, $\partial_3\varphi^2(z_3)$, leads to the same displacement solutions as those obtained by solving the P^0_{cell} problem. Thus, besides $\mathbf{\acute{u}}^3_{\text{part}}$, the only new unknowns of the current problem are the solutions corresponding to the derivatives of the first order macroscopic strains defined in Eq. (24). Consequently, the displacement field solution of the P^1_{cell} problem can be formally written as follows:

$$\mathbf{u}^3 = \mathbf{\acute{u}}^3(z_3,\mathbf{y}) + \boldsymbol{\chi}^1(\mathbf{y}) \cdot \mathbf{\acute{e}}^2(z_3) + \boldsymbol{\chi}^2(\mathbf{y}) \cdot \partial_3\mathbf{\acute{e}}^1(z_3) + \mathbf{\acute{u}}^3_{\text{part}}(z_3,\mathbf{y}) \tag{35}$$

where

$$\mathbf{\acute{u}}^3(z_3,\mathbf{y}) = \hat{u}_i^3(z_3) \cdot \mathbf{e}_{\mathbf{i}} + \varphi^3(z_3)[y_1 \cdot \mathbf{e}_{\mathbf{2}} - y_2 \cdot \mathbf{e}_{\mathbf{1}}] - y_\alpha \cdot \partial_3\hat{u}_\alpha^2(z_3) \cdot \mathbf{e}_{\mathbf{3}}, \tag{36}$$

$$\mathbf{\acute{e}}^2(z_3) = {}^{\mathrm{t}}\{\partial_3\hat{u}_3^2(z_3), \partial_{33}\hat{u}_1^1(z_3), \partial_{33}\hat{u}_2^1(z_3), \partial_3\varphi^2(z_3)\}, \tag{37}$$

$$\boldsymbol{\chi}^2(\underset{\sim}{\mathbf{y}}) = [\underset{\sim}{\boldsymbol{\chi}}^{2E}(\underset{\sim}{\mathbf{y}}), \underset{\sim}{\boldsymbol{\chi}}^{2C_1}(\underset{\sim}{\mathbf{y}}), \underset{\sim}{\boldsymbol{\chi}}^{2C_2}(\underset{\sim}{\mathbf{y}}), \underset{\sim}{\boldsymbol{\chi}}^{2T}(\underset{\sim}{\mathbf{y}})], \tag{38}$$

$$\partial_3 \acute{\underset{\sim}{\mathbf{e}}}^1(z_3) = {}^{\mathrm{t}}\{\partial_{33}\hat{u}_3^1(z_3), \partial_{333}\hat{u}_1^0(z_3), \partial_{333}\hat{u}_2^0(z_3), \partial_{33}\varphi^1(z_3)\}. \tag{39}$$

In expression (35), the fields grouped in $\boldsymbol{\chi}^2(\mathbf{y})$ are the solutions of the problem (31) with the data contained in $\partial_3\acute{\mathbf{e}}^1(z_3)$ as the only nonzero data: i.e. the body forces $\mathbf{h_1}(\partial_3\acute{\mathbf{e}}^1)$, no traction on ∂Y_b, and an initial strain state which is restricted to the last matrix of Eq. (33). The vector $\acute{\mathbf{e}}^2(z_3)$ stands for the second order macroscopic strains. The fields $\boldsymbol{\chi}^1(\mathbf{y})$ have already been defined in Section 4.3.

Remark 4.3. The field $\boldsymbol{\chi}^2$ has been introduced in Trabucho and Viaño (1996), Duva and Simmonds (1991) and Fan and Widera (1990) for beams with constant cross-section. In the homogeneous and isotropic case, analytical solution is available for $\boldsymbol{\chi}^{2E}$, and for $\boldsymbol{\chi}^{2C_\alpha}$ for some cross-sections. For heterogeneous and periodic beams, $\boldsymbol{\chi}^{2E}$ and $\boldsymbol{\chi}^{2T}$ appear in Kolpakov (1995). See also some related work in the case of periodic plates in Lewiński (1991a).

The stress field $\boldsymbol{\sigma}^2$ solution of the P_{cell}^1 problem can also be formally expressed as follows:

$$\begin{aligned}
\boldsymbol{\sigma}^2 &= [\boldsymbol{\tau}^{1E}(\underset{\sim}{\mathbf{y}}) \cdot \partial_3\hat{u}_3^2(z_3) + \boldsymbol{\tau}^{1C_\alpha}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{33}\hat{u}_\alpha^1(z_3) + \boldsymbol{\tau}^{1T}(\underset{\sim}{\mathbf{y}}) \cdot \partial_3\varphi^2(z_3)] + [\boldsymbol{\tau}^{2E}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{33}\hat{u}_3^1(z_3) \\
&\quad + \boldsymbol{\tau}^{2C_\alpha}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{333}\hat{u}_\alpha^0(z_3) + \boldsymbol{\tau}^{2T}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{33}\varphi^1(z_3)] + \acute{\boldsymbol{\tau}}_{\mathrm{part}}^2(z_3, \underset{\sim}{\mathbf{y}}) \equiv \boldsymbol{\tau}^1(\underset{\sim}{\mathbf{y}}) \cdot \acute{\underset{\sim}{\mathbf{e}}}^2(z_3) + \boldsymbol{\tau}^2(\underset{\sim}{\mathbf{y}}) \cdot \partial_3\acute{\underset{\sim}{\mathbf{e}}}^1(z_3) + \acute{\boldsymbol{\tau}}_{\mathrm{part}}^2(z_3, \underset{\sim}{\mathbf{y}})
\end{aligned} \tag{40}$$

with

$$\tau_{ij}^{2\bullet} = a_{ijk3}\chi_k^{1\bullet} + a_{ijkh}\partial_{y_h}\chi_k^{2\bullet} \tag{41}$$

The stress fields contained in the first brackets have been determined by solving the P_{cell}^0 problem, while those in the second brackets are four new elementary solutions of the P_{cell}^1 problem, when the four data of $\partial_3\acute{\mathbf{e}}^1(z_3)$ are prescribed. $\acute{\boldsymbol{\tau}}_{\mathrm{part}}^2$ is given by $\acute{\boldsymbol{\tau}}_{\mathrm{part}}^2 = \mathbf{a}(\underset{\sim}{\mathbf{y}}):\mathbf{grad}_{\mathbf{y}}^s(\acute{\underset{\sim}{\mathbf{u}}}_{\mathrm{part}}^3)$.

4.5. Generalization: formal expression of the outer displacement field

By now, we have gone far enough to see how to proceed the formal construction of the displacement field $\mathbf{u}^\varepsilon$.

Inserting Eq. (35) and Eq. (40) in the equations of the P_{cell}^2 problem, it is not difficult to see that the macroscopic data of this cellular problem will involve the third order macroscopic strains $\acute{\mathbf{e}}^3(z_3)$, the first gradient of the second order macroscopic strains (i.e. $\partial_3\acute{\mathbf{e}}^2(z_3)$), plus the second gradient of the first order macroscopic strains (i.e. $\partial_{33}\acute{\mathbf{e}}^1(z_3)$). Furthermore, the loadings $f_\alpha\mathbf{e}_\alpha$ and $g_\alpha\mathbf{e}_\alpha$ have to be added to these data, according to assumptions (6).

In a recursive manner, the number of data involved in a cellular problem P_{cell}^k will increase, starting from the $(k+1)$th order macroscopic strains $\acute{\mathbf{e}}^{k+1}(z_3)$ until the kth gradient of the first order macroscopic strains, $\partial_3^k\acute{\mathbf{e}}^1(z_3)$.

Thus, assuming that the data of each cellular problem verify the compatibility condition (14), the asymptotic expansion of the displacement field $\mathbf{u}^\varepsilon$ takes the form:

$$\underset{\sim}{\mathbf{u}}^{\varepsilon} = \hat{u}_{\alpha}^{0}(z_{3})\underset{\sim}{\mathbf{e}}_{\alpha} + \varepsilon^{1}\left[\underset{\sim}{\acute{\mathbf{u}}}^{1}(z_{3},\underset{\sim}{\mathbf{y}})\right] + \varepsilon^{2}\left[\underset{\sim}{\acute{\mathbf{u}}}^{2}(z_{3},\underset{\sim}{\mathbf{y}}) + \underset{\sim}{\boldsymbol{\chi}}^{1}(\underset{\sim}{\mathbf{y}}) \cdot \underset{\sim}{\acute{\mathbf{e}}}^{1}(z_{3})\right] + \varepsilon^{3}\left[\underset{\sim}{\acute{\mathbf{u}}}^{3}(z_{3},\underset{\sim}{\mathbf{y}}) + \underset{\sim}{\boldsymbol{\chi}}^{1}(\underset{\sim}{\mathbf{y}}) \cdot \underset{\sim}{\acute{\mathbf{e}}}^{2}(z_{3})\right.$$

$$\left. + \underset{\sim}{\boldsymbol{\chi}}^{2}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{3}\underset{\sim}{\acute{\mathbf{e}}}^{1}(z_{3}) + \underset{\sim}{\acute{\mathbf{u}}}^{3}_{\mathrm{part}}\right] + \varepsilon^{4}\left[\underset{\sim}{\acute{\mathbf{u}}}^{4}(z_{3},\underset{\sim}{\mathbf{y}}) + \underset{\sim}{\boldsymbol{\chi}}^{1}(\underset{\sim}{\mathbf{y}}) \cdot \underset{\sim}{\acute{\mathbf{e}}}^{3}(z_{3}) + \underset{\sim}{\boldsymbol{\chi}}^{2}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{3}\underset{\sim}{\acute{\mathbf{e}}}^{2}(z_{3})\right.$$

$$\left. + \underset{\sim}{\boldsymbol{\chi}}^{3}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{33}\underset{\sim}{\acute{\mathbf{e}}}^{1}(z_{3}) + \underset{\sim}{\acute{\mathbf{u}}}^{4}_{\mathrm{part}}\right] + \varepsilon^{5}[\cdots] \tag{42}$$

The expression (42) is similar to the asymptotic expansion obtained in the case of 3D periodic media (see relation (21) in Gambin and Kröner (1989)).

One can also express the asymptotic expansion of the stress field $\underset{\sim}{\boldsymbol{\sigma}}^{\varepsilon}$ under a similar recursive form. The generalization of expression (40) leads also to the following expansion:

$$\underset{\sim}{\boldsymbol{\sigma}}^{\varepsilon} = \varepsilon^{1}[\underset{\sim}{\boldsymbol{\tau}}^{1}(\underset{\sim}{\mathbf{y}}) \cdot \underset{\sim}{\acute{\mathbf{e}}}^{1}(z_{3})] + \varepsilon^{2}[\underset{\sim}{\boldsymbol{\tau}}^{1}(\underset{\sim}{\mathbf{y}}) \cdot \underset{\sim}{\acute{\mathbf{e}}}^{2}(z_{3}) + \underset{\sim}{\boldsymbol{\tau}}^{2}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{3}\underset{\sim}{\acute{\mathbf{e}}}^{1}(z_{3}) + \underset{\sim}{\acute{\boldsymbol{\tau}}}^{2}_{\mathrm{part}}] + \varepsilon^{3}[\underset{\sim}{\boldsymbol{\tau}}^{1}(\underset{\sim}{\mathbf{y}}) \cdot \underset{\sim}{\acute{\mathbf{e}}}^{3}(z_{3})$$

$$+ \underset{\sim}{\boldsymbol{\tau}}^{2}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{3}\underset{\sim}{\acute{\mathbf{e}}}^{2}(z_{3}) + \underset{\sim}{\boldsymbol{\tau}}^{3}(\underset{\sim}{\mathbf{y}}) \cdot \partial_{33}\underset{\sim}{\acute{\mathbf{e}}}^{1}(z_{3}) + \underset{\sim}{\acute{\boldsymbol{\tau}}}^{3}_{\mathrm{part}}] + \varepsilon^{4}[\cdots] \tag{43}$$

Relation (42) (and consequently Eq. (43)) constitutes a formal expression of the solution field in the sense that, by now, only the microscopic parts $\underset{\sim}{\boldsymbol{\chi}}^{i}(\underset{\sim}{\mathbf{y}})$ have been determined by solving in series the cellular problems. The macroscopic part of Eq. (42), characterized by the fields $\underset{\sim}{\acute{\mathbf{u}}}^{i}$ as well as their successive gradients, has now to be found. The way of obtaining it will be explained in the Section 5.

Remark 4.4. As in the treatment of the P_{cell}^{1} problem, it is necessary to take into account the compatibility conditions of the P_{cell}^{k} problem. Therefore, one has to solve P_{cell}^{k} in a similar way as made in the case of P_{cell}^{1} (see Sections 4.4 and 5.1.5). In that way, the elementary problems corresponding to each data of P_{cell}^{k} are well posed.

5. The set of macroscopic homogenized problems P_{hom}^{k}

As already mentioned in Section 4.1, the equilibrium equations corresponding to the unknown displacement fields $\underset{\sim}{\acute{\mathbf{u}}}^{i}$ are obtained from the compatibility condition (14). Expressing this condition for the cellular problems P_{cell}^{k} and P_{cell}^{k+1} leads indeed to the formulation of the homogenized 1D problems P_{hom}^{k}. This process will be applied in the next subsections: the way of deriving the equations of the first homogenized problem, denoted by P_{hom}^{1}, will be developed in detail in Section 5.1. A generalization will then be outlined in Section 5.2 in order to give the form of the general homogenized problem P_{hom}^{k}, with k corresponding to an arbitrary power of ε.

5.1. Formulation of the first homogenized problem P_{hom}^{1}

5.1.1. Equilibrium equations

Firstly, it must be noted that the compatibility condition (14) is satisfied identically for the first two cellular problems (16) and (20). As a consequence, the first homogenized problem occurs at order $k = 1$.

Let us first focus our attention on the derivation of the macroscopic equilibrium equations of the first homogenized problem P_{hom}^{1}.

The P_{cell}^{1} cellular problem (31) admits a solution provided that the data $\mathbf{div}_{z_{3}} \underset{\sim}{\boldsymbol{\sigma}}^{1}$, $(0, 0, f_{3})$ and $(0, 0, g_{3})$ satisfy the relation (14). In particular, if we choose as test functions $\mathbf{v}$ the four 'elementary' functions of $\mathfrak{R}$: $\hat{v}_{3}(z_{3})\underset{\sim}{\mathbf{e}}_{3}$, $\hat{v}_{\alpha}(z_{3})\underset{\sim}{\mathbf{e}}_{\alpha}$ and $(y_{1}\underset{\sim}{\mathbf{e}}_{2} - y_{2}\underset{\sim}{\mathbf{e}}_{1})$, condition (14) leads to the four following equations:

$$\frac{\partial}{\partial z_3} \int_{Y^*} \sigma_{33}^1 \, dY + \int_{Y^*} f_3 \, dY + \int_{\partial Y_b} g_3 \, d\Gamma = 0 \tag{44}$$

$$\frac{\partial}{\partial z_3} \int_{Y^*} \sigma_{\alpha 3}^1 \, dY = 0 \tag{45}$$

$$\frac{\partial}{\partial z_3} \int_{Y^*} (y_1 \sigma_{23}^1 - y_2 \sigma_{13}^1) \, dY = 0 \tag{46}$$

Furthermore, putting $\underset{\sim}{\psi} = y_\alpha \mathbf{e_3}$ in the variational formulation of the P_{cell}^1 problem, given by relation (13) with $k = 1$, we have:

$$\int_{Y^*} \sigma_{\alpha 3}^2 \, dY = \int_{Y^*} \left[y_\alpha \frac{\partial \sigma_{33}^1}{\partial z_3} + y_\alpha f_3 \right] dY + \int_{\partial Y_b} y_\alpha g_3 \, d\Gamma, \quad \alpha = 1, 2 \tag{47}$$

In the same manner, if we express the condition (14) for the P_{cell}^2 problem, choosing now the two test functions $\hat{v}_\alpha(z_3)\underset{\sim}{\mathbf{e_\alpha}}$ with $\alpha = 1$ or 2, we obtain:

$$\frac{\partial}{\partial z_3} \int_{Y^*} \sigma_{\alpha 3}^2 \, dY + \int_{Y^*} f_\alpha \, dY + \int_{\partial Y_b} g_\alpha \, d\Gamma = 0, \quad \alpha = 1, 2 \tag{48}$$

Let us introduce the following notations:

$$\begin{aligned} N^1(z_3) &= \langle \sigma_{33}^1 \rangle, \quad T_\alpha^2(z_3) = \langle \sigma_{\alpha 3}^2 \rangle \\ M_\alpha^1(z_3) &= \langle -y_\alpha \sigma_{33}^1 \rangle, \quad M_3^1(z_3) = \langle (-y_2 \sigma_{13}^1 + y_1 \sigma_{23}^1) \rangle \end{aligned} \tag{49}$$

with $\langle \cdot \rangle \equiv (1/l_3) \int_{Y^*} dY$ and where l_3 stands for the scaled length of period Y (see Fig. (2)).

The beam stresses $N^1(z_3)$, $T_\alpha^2(z_3)$, $M_\alpha^1(z_3)$ and $M_3^1(z_3)$ respectively correspond to the first order macroscopic axial force, the second order transverse shearing forces, the first order bending moments and the first order twisting moment. They are simply the average of their local corresponding quantity over the period length.

Remark 5.1. The definition of the bending moments according to Eq. (49) do not obey the classical conventions used in strength of material. Following Eq. (49), $M_1^1(z_3)$ and $-M_2^1(z_3)$ are about the $\underset{\sim}{\mathbf{e_2}}$- and $\underset{\sim}{\mathbf{e_1}}$-axis respectively, (see Fig. 3).

With notations (49), it becomes obvious that relations (44), and (46)–(48) can be written, respectively, as:
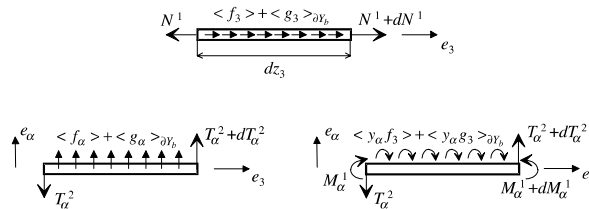

Fig. 3. Element of the equivalent beam (all loads and stress resultants are shown in their positive directions).

$$\partial_3 N^1 + \langle f_3 \rangle + \langle g_3 \rangle_{\partial Y_b} = 0$$

$$\partial_3 T_\alpha^2 + \langle f_\alpha \rangle + \langle g_\alpha \rangle_{\partial Y_b} = 0, \quad \alpha = 1, 2$$

$$-T_\alpha^2 - \partial_3 M_\alpha^1 + \langle y_\alpha f_3 \rangle + \langle y_\alpha g_3 \rangle_{\partial Y_b} = 0, \quad \alpha = 1, 2 \qquad (50)$$

$$\partial_3 M_3^1 = 0$$

with $\langle . \rangle_{\partial Y_b} \equiv (1/l_3) \int_{\partial Y_b} dY$.

Note that the set of the Eqs. (50) corresponds to the classical equilibrium relations of a beam theory problem and hence constitutes the local equations of the P_{hom}^1 problem. The Eqs. (50) (first, second and third equations) and beam stresses representation are illustrated Fig. 3, considering the equilibrium of a beam element of length dz_3.

Remark 5.2. It appears from Eq. (50) (third equation) that the first order bending moments M_α^1 are not related to the transverse shearing forces of the same order but to the second order ones T_α^2. As a matter of fact, the first order shearing resultants T_α^1 are equal to zero, and thus Eq. (45) is identically satisfied. This remarkable result can be easily established as follows: we first notice that T_α^1 can be defined as:

$$l_3 \cdot T_\alpha^1 = \int_{Y^*} \sigma_{\alpha 3}^1 \, dY \equiv \int_{Y^*} \boldsymbol{\sigma}^1 : \mathbf{grad}_y^s(\underset{\sim}{\psi}) \, dY \quad \text{with } \underset{\sim}{\psi} = y_\alpha \mathbf{e_3}$$

Green's formula can then be applied, so that:

$$l_3 \cdot T_\alpha^1 = \int_{\partial Y^*} \sigma_{3j}^1 n_j y_\alpha \, dS - \int_{Y^*} \frac{\partial \sigma_{3j}^1}{\partial y_j} y_\alpha \, dY \qquad (51)$$

where ∂Y^* is constituted of the lateral boundaries ∂Y_b and ∂Y_c and of the left and right sides of the period.

The first integral in Eq. (51) vanishes by virtue of the y_3-periodicity of $\boldsymbol{\sigma}^1$ and y_α, of the absence of prescribed surface force at this order on the lateral outer boundary ∂Y_b, and of the stress-free condition on the holes boundary ∂Y_c. In the same way, the second integral vanishes too, according to the equilibrium equation of the P_{cell}^0 problem.

5.1.2. Constitutive relations

After obtaining the equilibrium equations of the P_{hom}^1 problem, we focus now our attention on the constitutive stress–strain relations of P_{hom}^1.

Grouping the 'effective' (i.e. nonequal to zero) first order macroscopic beam stresses in a vector, the constitutive relations of the P_{hom}^1 problem can be defined as:

$$\begin{Bmatrix} N^1(z_3) \\ M_1^1(z_3) \\ M_2^1(z_3) \\ M_3^1(z_3) \end{Bmatrix} = \mathscr{A}^{\text{hom1}} \cdot \begin{Bmatrix} \partial_3 \hat{u}_3^1(z_3) \\ \partial_{33} \hat{u}_1^0(z_3) \\ \partial_{33} \hat{u}_2^0(z_3) \\ \partial_3 \varphi^1(z_3) \end{Bmatrix} \qquad (52)$$

and for later consistency of notations, relation (52) will be written in the form:

$$\underset{\sim}{\hat{\boldsymbol{\sigma}}}^1 = \mathscr{A}^{\text{hom1}} \cdot \underset{\sim}{\hat{\mathbf{e}}}^1 \qquad (53)$$

The components of the 4×4 matrix $\mathscr{A}^{\text{hom1}}$ are defined as follows:

$$\mathcal{A}^{\text{hom1}}_{1m} = \langle \tau^1_{33m} \rangle, \qquad \mathcal{A}^{\text{hom1}}_{2m} = \langle -y_1 \tau^1_{33m} \rangle$$
$$\mathcal{A}^{\text{hom1}}_{3m} = \langle -y_2 \tau^1_{33m} \rangle, \qquad \mathcal{A}^{\text{hom1}}_{4m} = \langle -y_2 \tau^1_{13m} + y_1 \tau^1_{23m} \rangle \tag{54}$$

with $1 \leqslant m \leqslant 4$. The quantities τ^1_{ijm}, obtained after solution of the P^0_{cell} cellular problem, have been defined in Eq. (30).

Consequently, $\mathcal{A}^{\text{hom1}}_{11}$ is the stretching stiffness, $\mathcal{A}^{\text{hom1}}_{22}$ and $\mathcal{A}^{\text{hom1}}_{33}$ the two bending stiffnesses, $\mathcal{A}^{\text{hom1}}_{44}$ the twisting stiffness and the extra-diagonal quantities are the different coupling terms. Note that the effective stiffness matrix $\mathcal{A}^{\text{hom1}}$ is determined from the solution of the first order cellular problem.

Following Sanchez-Hubert and Sanchez-Palencia (1992) for transversely nonhomogeneous rods or Caillerie (1984) for periodic plates, it can be proved that $\mathcal{A}^{\text{hom1}}$ fulfills the symmetry conditions $\mathcal{A}^{\text{hom1}}_{mn} = \mathcal{A}^{\text{hom1}}_{nm}$ and is positive definite.

5.1.3. Boundary conditions

To complete the formulation of the P^1_{hom} problem, it still remains to give the boundary conditions corresponding to the both ends $z_3 = 0, L$. More precisely, one has to derive from the exact boundary conditions expressed on the 2D end sections, S^ε_0 and S^ε_L, the prescribed data for the first order macroscopic functions for $z_3 = 0$ and $z_3 = L$.

As a first step, let us deal with the clamped condition on S^ε_0. Writing the boundary conditions $u^\varepsilon_i = 0$ at each power of ε leads to:

$$\underset{\sim}{\mathbf{u}}^m(0, \underset{\sim}{\mathbf{y}}) = \underset{\sim}{\mathbf{0}}, \quad m \geqslant 0 \tag{55}$$

which yields to the following conditions on the first two terms of the asymptotic expansion:

$$\hat{u}^0_\alpha(0) = 0 \quad \text{and} \quad \begin{cases} \hat{u}^1_1(0) - y_2 \varphi^1(0) = 0 \\ \hat{u}^1_2(0) + y_1 \varphi^1(0) = 0 \quad \forall y_\alpha. \\ \hat{u}^1_3(0) - y_\alpha \partial_3 \hat{u}^0_\alpha(0) = 0 \end{cases} \tag{56}$$

Thus, relations (56) can be identically satisfied provided that:

$$\hat{u}^0_\alpha(0) = \partial_3 \hat{u}^0_\alpha(0) = \hat{u}^1_3(0) = \varphi^1(0) = 0 \tag{57}$$

Relation (57) hence corresponds to the displacement boundary conditions of the first order homogenized problem P^1_{hom}.

Let us deal now with the other end section S^ε_L.

Firstly, recalling that the initial 3D conditions are $\sigma^\varepsilon_{i3} = \bar{\sigma}^\varepsilon_{i3}(x_1, x_2)$, and taking into account the order of magnitude of the prescribed stress data (6) yields:

$$\begin{cases} \sigma^1_{\alpha 3} = 0 & \sigma^2_{\alpha 3} = \bar{\sigma}_{\alpha 3}(y_1, y_2) \\ \sigma^1_{33} = \bar{\sigma}_{33}(y_1, y_2) & \sigma^2_{33} = 0 \end{cases} \quad \sigma^k_{i3} = 0 \ \ k \geqslant 2 \tag{58}$$

However, the stresses $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$ obtained from the cellular problems depend on the microscopic variables y_α, and are not able to satisfy arbitrary prescribed edge data $\bar{\sigma}_{i3}(y_1, y_2)$.

Therefore, a specific study is necessary in order to derive the appropriate boundary conditions on this end section. This will be treated in Part II of this paper, in which a rigorous justification of Saint–Venant's principle is provided. The initial 3D boundary conditions are thus written as:

$$\begin{cases} \int_{S^\varepsilon_L} \sigma^\varepsilon_{i3} \, dS^\varepsilon_L = \int_{S^\varepsilon_L} \bar{\sigma}^\varepsilon_{i3} \, dS^\varepsilon_L \\ \int_{S^\varepsilon_L} x_\alpha \underset{\sim}{\mathbf{e}_\alpha} \wedge \sigma^\varepsilon_{i3} \underset{\sim}{\mathbf{e}_i} \, dS^\varepsilon_L = \int_{S^\varepsilon_L} x_\alpha \underset{\sim}{\mathbf{e}_\alpha} \wedge \bar{\sigma}^\varepsilon_{i3} \underset{\sim}{\mathbf{e}_i} \, dS^\varepsilon_L \end{cases} \tag{59}$$

For the P_{hom}^1 problem, the boundary conditions have to be expressed as a function of the macroscopic stresses $\{N^1, T_\alpha^2, \underset{\sim}{\mathbf{M}}^1\}$, and Eq. (59) leads to:

$$\begin{cases} N^1(L) = \int_{S_L} \bar{\sigma}_{33} \, \mathrm{d}S_L, \quad T_\alpha^2(L) = \int_{S_L} \bar{\sigma}_{\alpha 3} \, \mathrm{d}S_L \\ M_\alpha^1(L) = \int_{S_L} -y_\alpha \bar{\sigma}_{33} \, \mathrm{d}S_L, \quad M_3^1(L) = 0 \end{cases} \tag{60}$$

The proof of Eq. (60) is based on the property that the macroscopic stresses $\acute{\boldsymbol{\sigma}}^1$ and T_α^2, which are obtained following an average process over the period Y, are also equal to the resultant beam forces on the right side of the period. Hence the boundary conditions (60), assuming that the structure is constituted of a whole number of periods. The relations (60) have been proposed in Cimetière et al. (1988), and Trabucho and Viaño (1996).

Remark 5.3. It can be seen in Eq. (60) that there is no torque applied to the end-section for the P_{hom}^1 problem. This result follows directly from the assumption $\bar{\sigma}_{\alpha 3}^\varepsilon(x_1, x_2) = \varepsilon^2 \cdot \bar{\sigma}_{\alpha 3}(y_1, y_2)$, which produces a torque at the second order. Indeed, this last assumption has been made in order to lead to a zero first order shearing force at the beam end, which is compatible with the result $T_\alpha^1 = 0$. However, this assumption might be relaxed and one might consider a distribution of $\bar{\sigma}_{\alpha 3}^\varepsilon$, such that the resultant shearing force remains zero but now with a nonzero resultant torque, so that $M_3^1 \neq 0$.

5.1.4. Summary

To summarize, the first order homogenized problem, P_{hom}^1, consists in finding the macroscopic stresses $\{N^1, T_\alpha^2, \underset{\sim}{\mathbf{M}}^1\}$ and the four macroscopic displacements $\{\hat{u}_\alpha^0, \hat{u}_3^1, \varphi^1\}$ such that:

$$\begin{cases} \partial_3 N^1 + \langle f_3 \rangle + \langle g_3 \rangle_{\partial Y_b} = 0 \\ \partial_3 T_\alpha^2 + \langle f_\alpha \rangle + \langle g_\alpha \rangle_{\partial Y_b} = 0 \\ -T_\alpha^2 + \partial_3 M_\alpha^1 + \langle y_\alpha f_3 \rangle + \langle y_\alpha g_3 \rangle_{\partial Y_b} = 0 \\ \partial_3 M_3^1 = 0 \\ \begin{Bmatrix} N^1(z_3) \\ M_1^1(z_3) \\ M_2^1(z_3) \\ M_3^1(z_3) \end{Bmatrix} = \mathscr{A}^{\text{hom}1} \cdot \begin{Bmatrix} \partial_3 \hat{u}_3^1(z_3) \\ \partial_{33} \hat{u}_1^0(z_3) \\ \partial_{33} \hat{u}_2^0(z_3) \\ \partial_3 \varphi^1(z_3) \end{Bmatrix} \\ \hat{u}_\alpha^0(0) = \partial_3 \hat{u}_\alpha^0(0) = \hat{u}_3^1(0) = \varphi^1(0) = 0 \\ N^1(L) = \int_{S_L} \bar{\sigma}_{33} \, \mathrm{d}S_L, \quad T_\alpha^2(L) = \int_{S_L} \bar{\sigma}_{\alpha 3} \, \mathrm{d}S_L \\ M_\alpha^1(L) = \int_{S_L} -y_\alpha \bar{\sigma}_{33} \, \mathrm{d}S_L, \quad M_3^1(L) = 0 \end{cases} \tag{61}$$

Due to the positive-definiteness of $\mathscr{A}^{\text{hom}1}$, it can be proved that the problem (61) is well posed.

The P_{hom}^1 problem (61) generalizes and justifies the Euler–Bernoulli–Navier's beam model, initially proposed for homogeneous isotropic rods. In the case of periodic heterogeneity, a coupled stretching–bending–torsion model is generally obtained. Its mathematical justification, using convergence results, can be found in Kolpakov (1991).

It must be noticed that the equilibrium equation for the torque with the boundary condition at the end $z_3 = L$ leads to $M_3^1(z_3) = 0$. Thus, if the torsion is not coupled with stretching or bending, $\varphi^1(z_3) = 0$ due to the boundary condition at $z_3 = 0$.

Note also that the P_{hom}^1 problem is a 1D beam problem that can easily be solved analytically. Only the construction of the constitutive matrix $\mathscr{A}^{\text{hom}1}$ requires generally a numerical solution of the cellular problem P_{cell}^0 posed on the period Y.

5.1.5. Treatment of the equilibrium equation of the P^1_{cell} problem

Since the relations between the macroscopic quantities $\acute{\sigma}^1$ and $\acute{e}^1$ are established, let us now come back to the P^1_{cell} cellular problem. As already explained in Section 4.4, one has to take into account the compatibility conditions when solving this problem, and the way of proceeding will be presented here.

Expressing the fictive volume force involved in P^1_{cell} in terms of the quantities $\partial_3 \acute{e}^1_p, p \in [1,4]$, the equilibrium equation (31) (first equation) yields:

$$\underline{\mathbf{div}}_{\mathbf{y}} \, \sigma^2 + \tau^1_{i3p}(\mathbf{y}) \partial_3 \acute{e}^1_p(z_3) \mathbf{e_i} + f_3 \mathbf{e_3} = \mathbf{0} \tag{62}$$

Furthermore, recalling that $\mathscr{A}^{\mathrm{hom1}}$ is positive definite, the stress–strain relation (53) can be written as:

$$\partial_3 \acute{\mathbf{e}}^1(z_3) = \mathscr{S}^{\mathrm{hom1}} \cdot \partial_3 \acute{\sigma}^1(z_3) \tag{63}$$

where $\mathscr{S}^{\mathrm{hom1}}$ denotes the inverse matrix of $\mathscr{A}^{\mathrm{hom1}}$.

As proved in Section 5.1.1, the compatibility conditions for the P^1_{cell} problem reduce to Eq. (50) (first and fourth equation). Inserting them into Eq. (63), we get:

$$\partial_3 \acute{e}^1_p(z_3) = -\mathscr{S}^{\mathrm{hom1}}_{p1}(\langle f_3 \rangle + \langle g_3 \rangle_{\partial Y_b}) + \mathscr{S}^{\mathrm{hom1}}_{p2} \, \partial_3 M^1_1(z_3) + \mathscr{S}^{\mathrm{hom1}}_{p3} \, \partial_3 M^1_2(z_3) \tag{64}$$

As a consequence, the equilibrium equation (62) has to be written as:

$$\underline{\mathbf{div}}_{\mathbf{y}} \, \sigma^2 + f_3 \mathbf{e_3} - \tau^1_{i3p}(\mathbf{y}) \mathscr{S}^{\mathrm{hom1}}_{p1}(\langle f_3 \rangle + \langle g_3 \rangle_{\partial Y_b}) \mathbf{e_i} + \tau^1_{i3p}(\mathbf{y}) \{ \mathscr{S}^{\mathrm{hom1}}_{p2} \mathscr{A}^{\mathrm{hom1}}_{2m} + \mathscr{S}^{\mathrm{hom1}}_{p3} \mathscr{A}^{\mathrm{hom1}}_{3m} \} \partial_3 \acute{e}^1_m \mathbf{e_i} = \mathbf{0} \tag{65}$$

with summation on the repeated indices, $i \in [1,3]$ and $(p,m) \in [1,4]^2$.

Remark 5.4. The relation (65) gives the exact definition of the functions $\mathbf{h_1}$, $\mathbf{h_2}$, $\mathbf{h_3}$, introduced in Eq. (34) in Section 4.4. Thus, it is obvious that $\mathbf{h_2}(f_3) = -\tau^1_{i3p}(\mathbf{y}) \mathscr{S}^{\mathrm{hom1}}_{p1} \langle f_3 \rangle \mathbf{e_i}$, $\mathbf{h_3}(g_3) = -\tau^{1\sim}_{i3p}(\mathbf{y}) \mathscr{S}^{\mathrm{hom1}}_{p1} \langle g_3 \rangle_{\partial Y_b} \mathbf{e_i}$ and that $\underset{\sim}{\mathbf{h_1}}$ is given by the last terms of Eq. (65).

In the case of a constitutive law $\mathscr{A}^{\mathrm{hom1}}$ without any coupling, the relation (65) can be simplified in the following manner:

$$\underline{\mathbf{div}}_{\mathbf{y}} \, \sigma^2 + f_3 \mathbf{e_3} - \tau^{1E}_{i3}(\mathbf{y})(\mathscr{A}^{\mathrm{hom1}}_{11})^{-1}(\langle f_3 \rangle + \langle g_3 \rangle_{\partial Y_b}) \mathbf{e_i} + \tau^{1C_\alpha}_{i3}(\mathbf{y}) \partial_{333} \hat{u}^0_\alpha(z_3) \mathbf{e_i} = \underset{\sim}{0} \tag{66}$$

5.2. Formulation of the kth homogenized problem P^k_{hom}

5.2.1. Equilibrium equations

In the preceding section, it has been shown how to derive the first order homogenized problem P^1_{hom}. By applying exactly the same method for each order $k > 1$, one obtains the formulation of the higher-order homogenized problems, P^k_{hom}.

Hence the equilibrium equations of the P^k_{hom} problems:

$$\partial_3 N^k + \langle f^k_3 \rangle + \langle g^{k+1}_3 \rangle_{\partial Y_b} = 0 \tag{67}$$

$$\partial_3 T^{k+1}_\alpha + \langle f^{k+1}_\alpha \rangle + \langle g^{k+2}_\alpha \rangle_{\partial Y_b} = 0 \tag{68}$$

$$-T^{k+1}_\alpha - \partial_3 M^k_\alpha + \langle y_\alpha f^k_3 \rangle + \langle y_\alpha g^{k+1}_3 \rangle_{\partial Y_b} = 0 \tag{69}$$

$$\partial_3 M^k_3 + \langle y_1 f^k_2 - y_2 f^k_1 \rangle + \langle y_1 g^{k+1}_2 - y_2 g^{k+1}_1 \rangle_{\partial Y_b} = 0 \tag{70}$$

with

$$N^k(z_3) = \langle \sigma_{33}^k \rangle, \quad T_\alpha^{k+1}(z_3) = \langle \sigma_{\alpha 3}^{k+1} \rangle$$
$$M_\alpha^k(z_3) = \langle -y_\alpha \sigma_{33}^k \rangle, \quad M_3^k(z_3) = \langle (-y_2 \sigma_{13}^k + y_1 \sigma_{23}^k) \rangle \tag{71}$$

N^k, M_β^k and M_3^k correspond respectively to the macroscopic axial force, bending moments and twisting moment of order k, while T_α^{k+1} represent the macroscopic shearing forces of order $(k+1)$. We recall that the prescribed volume force $\underset{\sim}{\mathbf{f}}^k$ and surface force $\underset{\sim}{\mathbf{g}}^{k+1}$ satisfy assumptions (6).

5.2.2. Constitutive relations

Let us now study the macroscopic stress–strain relation of order k. We have seen in Eq. (43) that the stress field $\underset{\sim}{\sigma}^k$, solution of the P_{cell}^{k-1} problem, is a linear function of the kth order macroscopic strain $\underset{\sim}{\acute{\mathbf{e}}}^k$, the first-gradient of the $(k-1)$th order macroscopic strain (i.e. $\partial_3 \underset{\sim}{\acute{\mathbf{e}}}^{k-1}$), and so on until the $(k-1)$th-gradient of the first order macroscopic strain (i.e. $\partial_3^{k-1} \underset{\sim}{\acute{\mathbf{e}}}^1$). Therefore, the macroscopic stress–strain relation at any order k with $k \geqslant 1$ can be written as:

$$\underset{\sim}{\acute{\sigma}}^k = \mathscr{A}^{\mathrm{hom}1} \cdot \underset{\sim}{\acute{\mathbf{e}}}^k + \mathscr{A}^{\mathrm{hom}2} \cdot \partial_3 \underset{\sim}{\acute{\mathbf{e}}}^{k-1} + \mathscr{A}^{\mathrm{hom}3} \cdot \partial_{33} \underset{\sim}{\acute{\mathbf{e}}}^{k-2} + \cdots + \mathscr{A}^{\mathrm{hom}\,k} \cdot \partial_3^{k-1} \underset{\sim}{\acute{\mathbf{e}}}^1 + \underset{\sim}{\acute{\sigma}}^k_{\mathrm{part}}$$
$$\text{with} \quad \underset{\sim}{\acute{\sigma}}^k(z_3) = {}^{\mathrm{t}} \{ N^k, M_1^k, M_2^k, M_3^k \} \tag{72}$$

where strain vectors $\underset{\sim}{\acute{\mathbf{e}}}^p$ vanish when $p \leqslant 0$, and where ∂_3^{k-1} denotes the partial derivative $(\partial^{k-1}/\partial z_3^{k-1})$.

The kth order stress vector $\underset{\sim}{\acute{\sigma}}^k_{\mathrm{part}}$ contains the beam forces deduced from the stress state $\underset{\sim}{\acute{\tau}}^k_{\mathrm{part}}$, i.e. the particular solution of the well-posed P_{cell}^{k-1} problem. This solution is obtained considering as data the volume and surface forces involved in the current problem, if any, as well as the derivatives of the particular solutions obtained at the preceding orders, $\mathbf{grad}^{\mathbf{s}}_{z_3}(\underset{\sim}{\acute{\mathbf{u}}}^k_{\mathrm{part}})$ and $\mathbf{div}_{z_3}(\underset{\sim}{\acute{\tau}}^{k-1}_{\mathrm{part}})$. The 4×4 matrix $\mathscr{A}^{\mathrm{hom}1}$ has already been defined in Eq. (54). In a similar way, the components of the 4×4 matrix $\mathscr{A}^{\mathrm{hom}\,k}$ are deduced from the four elementary stress solutions of the $(k-1)$th order cellular problem, namely τ^{kE}, τ^{kC_α} and τ^{kT}, grouped in τ^k. We recall that these stress tensors correspond to the solution of P_{cell}^{k-1} when the components of the $(k-1)$th gradient of $\underset{\sim}{\acute{\mathbf{e}}}^1$ are respectively considered as data, i.e. $\partial_3^k \hat{u}_3^1(z_3)$, $\partial_3^{k+1} \hat{u}_\alpha^0(z_3)$, and $\partial_3^k \varphi^1(z_3)$. Thus, $\mathscr{A}^{\mathrm{hom}\,k}$ is defined as:

$$\mathscr{A}_{1m}^{\mathrm{hom}\,k} = \langle \tau_{33m}^k \rangle, \quad \mathscr{A}_{2m}^{\mathrm{hom}\,k} = \langle -y_1 \tau_{33m}^k \rangle$$
$$\mathscr{A}_{3m}^{\mathrm{hom}\,k} = \langle -y_2 \tau_{33m}^k \rangle, \quad \mathscr{A}_{4m}^{\mathrm{hom}\,k} = \langle -y_2 \tau_{13m}^k + y_1 \tau_{23m}^k \rangle \tag{73}$$

Contrary to the first order effective stiffness matrix $\mathscr{A}^{\mathrm{hom}1}$, the higher-order stress–strain matrices $\mathscr{A}^{\mathrm{hom}\,k}$, $k \geqslant 2$, are not necessarily symmetric or positive definite tensors. Especially, the second order one, $\mathscr{A}^{\mathrm{hom}2}$, appears to be antisymmetric in 3D periodic media and even equal to zero following certain symmetry properties of the period Y (Boutin, 1996; Triantafyllidis and Bardenhagen, 1996).

Remark 5.5. Solving the kth order homogenized problem P_{hom}^k implies that the lower-order macroscopic problems have already been solved. Therefore, when considering the P_{hom}^k problem, the macroscopic strains $\underset{\sim}{\acute{\mathbf{e}}}^1, \ldots, \underset{\sim}{\acute{\mathbf{e}}}^{k-1}$ are known and so are their successive gradients. Therefore, the only unknown strain field in the right-hand side of relation (72) is the kth order strain vector $\underset{\sim}{\acute{\mathbf{e}}}^k$. All the other terms constitute data for the P_{hom}^k problem and can be considered as fictive initial stress states for the current macroscopic problem.

As a matter of fact, following the method presented in Boutin (1996), the first equilibrium equation (67) of the P_{hom}^k problem may be written as:

$$\partial_3(\mathscr{A}_{1m}^{\mathrm{hom}1} \cdot \acute{e}_m^k) = -\langle f_3^k \rangle - \langle g_3^{k+1} \rangle_{\partial Y_b} - \partial_3(\mathscr{A}_{1m}^{\mathrm{hom}2} \cdot \partial_3 \acute{e}_m^{k-1} + \mathscr{A}_{1m}^{\mathrm{hom}3} \cdot \partial_{33} \acute{e}_m^{k-2} + \cdots) \quad \text{with } 1 \leqslant m \leqslant 4. \tag{74}$$

Writing all the equilibrium equations of P_{hom}^k in a similar way shows that this problem may be regarded as the coupled stretching–bending–torsion model of Section 5.1.4, the higher-order effects arising under the form of fictive volume loadings. It becomes also clear that the displacement unknowns of the P_{hom}^k problem are the four macroscopic quantities $\{\hat{u}_\alpha^{k-1}, \hat{u}_3^k, \varphi^k\}$.

5.2.3. Boundary conditions

To complete the formulation of the P_{hom}^k problem, one must add to equilibrium equations (67)–(70) and constitutive relations (72) the boundary conditions which have to be expressed on the displacements $\{\hat{u}_\alpha^{k-1}, \hat{u}_3^k, \varphi^k\}$ for $z_3 = 0$ on one hand, and on the kth order macroscopic stresses $\{N^k, T_\alpha^{k+1}, \underset{\sim}{\mathbf{M}}^k\}$ for $z_3 = L$ on the other hand.

These conditions, which are obtained from the initial 3D conditions on the two end sections S_0^ε, S_L^ε, are given in Eqs. (55) and (58). Since we are interested in the P_{hom}^k problem with $k \geqslant 2$, it can be seen from Eqs. (42) and (43) that it is impossible to fulfill these conditions exactly, so that boundary layers arise at the two ends of the beam. This is a classical problem in asymptotic analysis of slender structures. In Part II of this paper, a method is proposed to derive the macroscopic boundary conditions at each order, so that well-posed P_{hom}^k problems are obtained.

6. Summary

Let us summarize here the results provided by the formal asymptotic method. The solution in series of the first k cellular problems, P_{cell}^{-1} to P_{cell}^{k-1}, leads to the determination of the y_3-periodic displacement fields $\{\underset{\sim}{\boldsymbol{\chi}}^1(\mathbf{y}), \ldots, \underset{\sim}{\boldsymbol{\chi}}^k(\mathbf{y})\}$, as well as the associated periodic stress fields $\{\boldsymbol{\tau}^{kE}, \boldsymbol{\tau}^{kC_\alpha}, \boldsymbol{\tau}^{kT}\}$.

Then, following the average process given in Eq. (73), the first k effective matrices $\mathscr{A}^{\text{hom } k}$ can be calculated.

Treating in parallel the first k macroscopic problems, P_{hom}^1 to P_{hom}^k, leads to the macroscopic parts of the asymptotic expansions (42) and (43). Especially, the solution of the homogenized problems up to the kth order, P_{hom}^k, gives the macroscopic axial displacement $\hat{u}_3^p(z_3)$ and the macroscopic torsion rotation $\varphi^p(z_3)$ up to order k, as well as the macroscopic deflections $\hat{u}_\alpha^p(z_3)$ up to order $(k-1)$.

Thus, after having solved in series the cellular and the homogenized problems, one obtains from Eqs. (42) and (43), both local and global information on the solutions $\mathbf{u}^\varepsilon$, $\underset{\sim}{\boldsymbol{\sigma}}^\varepsilon$ of the initial problem. Particularly, the macroscopic description of the displacements of the structure is given by:

$$
\begin{aligned}
\hat{u}_\alpha(z_3) &= \hat{u}_\alpha^0(z_3) + \varepsilon \hat{u}_\alpha^1(z_3) + \cdots \\
\hat{u}_3(z_3) &= \hat{u}_3^1(z_3) + \varepsilon \hat{u}_3^2(z_3) + \cdots \\
\varphi(z_3) &= \varphi^1(z_3) + \varepsilon \varphi^2(z_3) + \cdots
\end{aligned}
\tag{75}
$$

7. Concluding remarks

In this paper, it is shown that the asymptotic expansion method provides a rigorous and systematic way to derive the overall response of a periodic heterogeneous beam. Especially, the macroscopic description of the displacement field is given by $\underset{\sim}{\acute{\mathbf{u}}}^\varepsilon$, defined as:

$$\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}(z_3, \underset{\sim}{\mathbf{y}}) = \underset{\sim}{\mathbf{u}}^0(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon \acute{\underset{\sim}{\mathbf{u}}}^1(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon^2 \acute{\underset{\sim}{\mathbf{u}}}^2(z_3, \underset{\sim}{\mathbf{y}}) + \cdots \tag{76}$$

so that the components of $\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}$ are given by:

$$\begin{cases} \acute{u}_1^{\varepsilon}(z_3, \underset{\sim}{\mathbf{y}}) = \hat{u}_1(z_3) - \varepsilon y_2 \varphi(z_3) \\ \acute{u}_2^{\varepsilon}(z_3, \underset{\sim}{\mathbf{y}}) = \hat{u}_2(z_3) + \varepsilon y_1 \varphi(z_3) \\ \acute{u}_3^{\varepsilon}(z_3, \underset{\sim}{\mathbf{y}}) = \varepsilon \hat{u}_3(z_3) - \varepsilon y_\alpha \partial_3 \hat{u}_\alpha(z_3) \end{cases} \tag{77}$$

with $\hat{u}_i(z_3)$ and $\varphi(z_3)$ defined in Eq. (75).

The determination of the global field $\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}$ may be achieved through a rational calculation of the successive terms of the interior expansions. Thus it is necessary to solve in series several 3D microscopic problems as well as 1D homogenized problems to find $\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}$ up to a certain desired order. The cellular problems allow us to characterize the beam response at the period scale, under different macroscopic loadings corresponding to macroscopic strains and their derivatives. Thus the effective beam behavior is obtained.

Nevertheless, it should be more judicious to define one homogenized problem which would enable us the derivation in a single step of $\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}$ up to the desired order. To this end, let us derive from the successive homogenized problems P_{hom}^k the 1D equations involving the unknowns $\hat{u}_i(z_3)$ and $\varphi(z_3)$ of $\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}$.

Introducing the field $\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}$ into the expansion (42), we see that the displacement field solution of the initial problem (3) can be written as:

$$\underset{\sim}{\mathbf{u}}^{\varepsilon}(z_3, \underset{\sim}{\mathbf{y}}) = \acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}(z_3, \underset{\sim}{\mathbf{y}}) + \varepsilon \underset{\sim}{\boldsymbol{\chi}}^1(\underset{\sim}{\mathbf{y}}) \cdot \acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) + \varepsilon^2 \underset{\sim}{\boldsymbol{\chi}}^2(\underset{\sim}{\mathbf{y}}) \cdot \partial_3 \acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) + \varepsilon^3 \underset{\sim}{\boldsymbol{\chi}}^3(\underset{\sim}{\mathbf{y}}) \cdot \partial_{33} \acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) + \cdots + \acute{\underset{\sim}{\mathbf{u}}}_{\mathrm{part}} \text{ with}$$

$$\acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) = \acute{\underset{\sim}{\mathbf{e}}}^1 + \varepsilon \acute{\underset{\sim}{\mathbf{e}}}^2 + \cdots = {}^{\mathrm{t}}\{\partial_3 \hat{u}_3, \partial_{33} \hat{u}_1, \partial_{33} \hat{u}_2, \partial_3 \varphi\} \text{ and } \acute{\underset{\sim}{\mathbf{u}}}_{\mathrm{part}} = \varepsilon^3 \acute{\underset{\sim}{\mathbf{u}}}_{\mathrm{part}}^3 + \varepsilon^4 \acute{\underset{\sim}{\mathbf{u}}}_{\mathrm{part}}^4 + \cdots \tag{78}$$

In the same way, Eq. (43) can be written as follows:

$$\boldsymbol{\sigma}^{\varepsilon} = \varepsilon \boldsymbol{\tau}^1 \cdot \acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) + \varepsilon^2 \boldsymbol{\tau}^2 \cdot \partial_3 \acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) + \varepsilon^3 \boldsymbol{\tau}^3 \cdot \partial_{33} \acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) + \cdots + \acute{\boldsymbol{\sigma}}_{\mathrm{part}} \tag{79}$$

Moreover, the expansion of the macroscopic beam stresses $\acute{\underset{\sim}{\boldsymbol{\sigma}}}^{\varepsilon}$ is defined by:

$$\acute{\underset{\sim}{\boldsymbol{\sigma}}}^{\varepsilon} = \acute{\underset{\sim}{\boldsymbol{\sigma}}}^1 + \varepsilon \acute{\underset{\sim}{\boldsymbol{\sigma}}}^2 + \cdots \tag{80}$$

and one has for the transverse shearing forces:

$$T_\alpha^{\varepsilon} = T_\alpha^2 + \varepsilon T_\alpha^3 + \cdots \tag{81}$$

Thus, from Eq. (72) and from the addition of the equilibrium equations at each order, the macroscopic fields $\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}$, $\acute{\underset{\sim}{\boldsymbol{\sigma}}}^{\varepsilon}$ and T_α^{ε} are found to satisfy:

$$\begin{cases} \partial_3 N^{\varepsilon} + \langle f_3 \rangle + \langle g_3 \rangle_{\partial Y_b} = 0 \\ \partial_3 T_\alpha^{\varepsilon} + \langle f_\alpha \rangle + \langle g_\alpha \rangle_{\partial Y_b} = 0 \\ -T_\alpha^{\varepsilon} - \partial_3 M_\alpha^{\varepsilon} + \langle y_\alpha f_3 \rangle + \langle y_\alpha g_3 \rangle_{\partial Y_b} = 0 \\ \partial_3 M_3^{\varepsilon} + \langle y_1 f_2 - y_2 f_1 \rangle + \langle y_1 g_2 - y_2 g_1 \rangle_{\partial Y_b} = 0 \end{cases} \tag{82}$$

$$\acute{\underset{\sim}{\boldsymbol{\sigma}}}^{\varepsilon} = {}^{\mathrm{t}}\{N^{\varepsilon}(z_3), M_1^{\varepsilon}(z_3), M_2^{\varepsilon}(z_3), M_3^{\varepsilon}(z_3)\}$$

$$= \mathscr{A}^{\mathrm{hom1}} \cdot \acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) + \varepsilon \mathscr{A}^{\mathrm{hom2}} \cdot \partial_3 \acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) + \varepsilon^2 \mathscr{A}^{\mathrm{hom3}} \cdot \partial_{33} \acute{\underset{\sim}{\mathbf{e}}}(\acute{\underset{\sim}{\mathbf{u}}}^{\varepsilon}) + \cdots + \acute{\underset{\sim}{\boldsymbol{\sigma}}}_{\mathrm{part}} \tag{83}$$

It is interesting to note that the macroscopic stress–strain relation (83) contains strain gradients up to infinite order. As a consequence, the macroscopic description obtained when taking into account higher-

order terms can be regarded as a higher-order gradient theory. This result was already pointed out in Gambin and Kröner (1989) in the case of 3D elastic periodic media. Indeed, for such problems, the asymptotic expansion method including higher-order terms brings out the contribution of nonlocal terms under the form of the successive strain gradients, exactly as in Eq. (83).

Furthermore, if we restrict the study to the first three terms of the expansion of $\grave{\underset{\sim}{\sigma}}^{\varepsilon}$, then the stress–strain relation (83) appears to generalize the well-known second gradient theory. Moreover if $\mathscr{A}^{\text{hom2}} = 0$ (which is obtained when the period Y^{ε} presents certain elastic symmetries), we recover exactly the latter theory. This comparison between the higher order theory derived from the asymptotic expansion method and the second gradient theory is widely discussed in (Boutin, 1996) for 3D periodic media.

Another interesting point of view is to draw a parallel between the global model given by Eqs. (82) and (83) and refined beam theories, i.e. more sophisticated 1D beam theories than Euler–Bernoulli's one. In that way, in the case of homogeneous isotropic rods, Timoshenko's model can be recovered and thus justified via the asymptotic expansion method. This justification is given in Fan and Widera (1990) or Trabucho and Viaño (1996), where a generalization of Timoshenko's theory is also established for the isotropic nonhomogeneous case. Let us outline here the way of proceeding to recover Timoshenko's theory from the general asymptotic model Eqs. (82) and (83). To this end, we consider the case of bending of a homogeneous isotropic rod in one of its principle planes ($\mathbf{e_1} - \mathbf{e_3}$), by the external forces $(f_1^{\varepsilon}, 0, 0)$ and $(g_1^{\varepsilon}, 0, 0)$ verifying (6), and with a clamping condition at the both ends S_0^{ε}, S_L^{ε}. Our aim is to derive the second order model associated to the approximation of the expansions $\grave{\underset{\sim}{\mathbf{u}}}^{\varepsilon}, \grave{\underset{\sim}{\sigma}}^{\varepsilon}$ up to the second nonzero term. The first two effective terms of the macroscopic beam forces are found to be $\grave{\underset{\sim}{\sigma}}^{\varepsilon} = \grave{\underset{\sim}{\sigma}}^{1} + \varepsilon^2 \grave{\underset{\sim}{\sigma}}^{3}$, and from Eq. (82),

$$\begin{cases} \partial_3(N^1 + \varepsilon^2 N^3) = 0 \\ \partial_3(T_1^2 + \varepsilon^2 T_1^4) + \int_S f_1 \, \mathrm{d}S + \int_{\partial S} g_1 \, \mathrm{d}\gamma = 0 \\ (T_1^2 + \varepsilon^2 T_1^4) + \partial_3(M_1^1 + \varepsilon^2 M_1^3) = 0 \end{cases} \tag{84}$$

since, for the beam under consideration here, the operators $\langle \cdot \rangle$ and $\langle \cdot \rangle_{\partial Y_b}$ can be reduced to $\int_S \cdot \, \mathrm{d}S$ and $\int_{\partial S} \cdot \, \mathrm{d}\gamma$, where S and ∂S stand for the scaled beam cross-section and its lateral boundary respectively.

Considering moreover the case where the bending does not give rise to either torsion or tension effects (symmetric bending without any coupling), and given that $\mathscr{A}^{\text{hom2}}$ is zero for a homogeneous beam, the stress–strain relation (83) leads to:

$$\begin{cases} M_1^1 = \mathscr{A}_{22}^{\text{hom1}} \, \partial_{33}\hat{u}_1^0 = EI_1 \, \partial_{33}\hat{u}_1^0 \\ M_1^3 = \mathscr{A}_{22}^{\text{hom1}} \, \partial_{33}\hat{u}_1^2 + \mathscr{A}_{22}^{\text{hom3}} \, \partial_3^4 \hat{u}_1^0 \end{cases} \tag{85}$$

with $I_1 = \int_S y_1^2 \, \mathrm{d}S$ and where E denotes the Young's modulus.

Consequently, the equilibrium equations (84) (second and third equations) yield:

$$\mathscr{A}_{22}^{\text{hom1}} \, \partial_3^4(\hat{u}_1^0 + \varepsilon^2 \hat{u}_1^2) + \varepsilon^2 \mathscr{A}_{22}^{\text{hom3}} \, \partial_3^6(\hat{u}_1^0) - \int_S f_1 \, \mathrm{d}S - \int_{\partial S} g_1 \, \mathrm{d}\gamma = 0 \tag{86}$$

Moreover, from the first-order homogenized problem, the deflection $\hat{u}_1^0$ is the solution of the differential equation $\mathscr{A}_{22}^{\text{hom1}} \, \partial_3^4(\hat{u}_1^0) + \int_S f_1 \, \mathrm{d}S + \int_{\partial S} g_1 \, \mathrm{d}\gamma = 0$, so that Eq. (86) can be written under the form:

$$\mathscr{A}_{22}^{\text{hom1}} \, \partial_3^4 \hat{u}_1^{2\varepsilon} + \varepsilon^2 \mathscr{A}_{22}^{\text{hom3}} (\mathscr{A}_{22}^{\text{hom1}})^{-1} \, \partial_3^2 \left(\int_S f_1 \, \mathrm{d}S + \int_{\partial S} g_1 \, \mathrm{d}\gamma \right) - \int_S f_1 \, \mathrm{d}S - \int_{\partial S} g_1 \, \mathrm{d}\gamma = 0 \tag{87}$$

with $\hat{u}_1^{2\varepsilon} = \hat{u}_1^0 + \varepsilon^2 \hat{u}_1^2$.

The relation (87) is found to be exactly of the form of the differential equation for Timoshenko's beam theory (u_1^{T} stands for the beam deflection):

$$EI_1\,\partial_3^4 u_1^{\mathrm{T}} + (kGS)^{-1}EI_1\,\partial_3^2\left(\int_S f_1\,\mathrm{d}S + \int_{\partial S} g_1\,\mathrm{d}\gamma\right) - \int_S f_1\,\mathrm{d}S - \int_{\partial S} g_1\,\mathrm{d}\gamma = 0 \tag{88}$$

where k denotes the shear correction factor introduced by Timoshenko.

Therefore, the comparison between Eqs. (87) and (88) enables us the definition of a shear coefficient k from $\mathscr{A}_{22}^{\mathrm{hom1}}$ and $\varepsilon^2 \mathscr{A}_{22}^{\mathrm{hom3}}$ (which corresponds to the descaled effective behavior). Nevertheless, it must be noticed that several terms in the asymptotic expansion (42) of the complete deflection $u_\alpha^\varepsilon(z_3, y_\alpha)$ have to be neglected in order that Eq. (87) reduces to Timoshenko's theory. Especially, Poisson's effects as well as geometrical torsional effects are neglected. The second order model (82) and (83) thus incorporates 3D effects which are not taken into account in the classical Timoshenko theory.

Generally speaking, the asymptotic expansion method has the advantage of taking into account, in a consistent and systematic way, nonclassical effects such as cross-sectional warping, as well as transverse shear and normal stresses and strains. This is a major difference from existing higher-order beam theories (see e.g. Kosmatka (1993), Reddy et al. (1997), Soldatos and Watson (1997) and references herein) which are based on a priori assumptions regarding stress and displacement variations. These theories are found to capture only a part of the correction due to higher-order effects, as it was proved previously for the second order Timoshenko theory.

Moreover, since approximate boundary conditions have to be considered, edge effects are an important source of errors in refined engineering theory (Duva and Simmonds, 1991). On the contrary, the asymptotic expansion method enables us to obtain an outer solution which is valid far from the edges (see Part II of this paper).

Acknowledgements

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